

# LIMIT LINEAR SERIES AND RANKS OF MULTIPLICATION MAPS

FU LIU, BRIAN OSSERMAN, MONTSERRAT TEIXIDOR I BIGAS,  
AND NAIZHEN ZHANG

ABSTRACT. We develop a new technique for studying ranks of multiplication maps for linear series via limit linear series and degenerations to chains of genus-1 curves. We use this approach to prove a purely elementary criterion for proving cases of the Maximal Rank Conjecture, and then apply the criterion to several ranges of cases, giving a new proof of the case of quadrics, and also treating several families in the case of cubics. Our proofs do not require restrictions on direction of approach, so we recover new information on the locus in the moduli space of curves on which the maximal rank condition fails.

## CONTENTS

1. Introduction	1
Structure	4
Acknowledgements	5
Conventions	5
2. Nondegeneracy on twice-marked elliptic curves	5
3. The elementary criterion: statement and examples	9
4. Proof of the elementary criterion	15
5. Observations on injectivity	26
6. The case of quadrics	29
7. Observations on surjectivity	34
8. The case of cubics	39
References	43

## 1. INTRODUCTION

The classical Brill-Noether theorem states that if we are given  $g, r, d \geq 0$ , a general curve  $X$  of genus  $g$  carries a linear series  $(\mathcal{L}, V)$  of rank  $r$  and degree  $d$  if and only if the quantity

$$\rho := g - (r + 1)(r + g - d)$$

is nonnegative [GH80]. Eisenbud and Harris proved that (at least in characteristic 0) when  $r \geq 3$ , a general such linear series on  $X$  will define an imbedding of  $X$  as a (nondegenerate) curve of degree  $d$  in  $\mathbb{P}^r$  [EH83a]. One of the most basic questions

---

Fu Liu is partially supported by NSF grant DMS-1265702. Brian Osserman is partially supported by a grant from the Simons Foundation #279151.

one might then ask is: what are the degrees of the equations defining  $X$ ? More precisely, we have the following question:

**Question 1.1.** In the above setting, for each  $m \geq 2$ , what is the dimension of the space of homogeneous polynomials of degree  $m$  vanishing on the image of  $X$ ?

Stated this way, the question is about the kernel of the natural restriction map

$$(1.1) \quad \Gamma(\mathbb{P}^r, \mathcal{O}(m)) \rightarrow \Gamma(X, \mathcal{L}^{\otimes m}).$$

The dimension of the source space is  $\binom{r+m}{m}$ , and it is a consequence of the Gieseker-Petri theorem that the dimension of the target space is  $md + 1 - g$ . The *Maximal Rank Conjecture* states that the rank of this map is as large as possible, or equivalently, the kernel of this map is as small as possible.

**Conjecture 1.2.** *The rank of the restriction map (1.1) is*

$$\min \left\{ \binom{r+m}{m}, md + 1 - g \right\},$$

so the answer to Question 1.1 is

$$\max \left\{ 0, \binom{r+m}{m} - (md + 1 - g) \right\}.$$

At least in part, this conjecture goes back to work of Noether in late 1800's, and of Severi in the early 1900's, but it was stated explicitly by Harris in 1982, and has received considerable attention since then. Partial results are due to Ballico and Ellia [Bal12b] [Bal12a] [Bal09] [BE87b] [BE87a], Voisin [Voi92], Farkas [Far09], Teixidor [Tei03], Larson [Lar12], and most recently, Jensen and Payne [JP16], [JP]. These results were in some cases motivated directly by the conjecture, but in other cases by a variety of applications, including to surjectivity of the Wahl map, to higher-rank Brill-Noether theory, and to the birational geometry of moduli spaces of curves. Subsequently, Aprodu and Farkas introduced a *Strong Maximal Rank Conjecture* motivated by applications to moduli spaces of curves [AF11], and Farkas and Ortega then developed the relationship to higher-rank Brill-Noether theory [FO11]. Taken together, the above-mentioned papers have treated Conjecture 1.2 in the following cases: when  $d \geq r + g$ ; when  $r = 3$  or  $r = 4$ ; when  $m = 2$ ; when  $d$  is sufficiently large relative to  $r$  and  $m$ ; and in several additional ranges of cases for  $m = 3$ , including many cases with  $r = 5$ . It is also important to note that if (1.1) is known to be surjective for a given  $m$  and a given linear series on a given curve, then surjectivity also follows for all larger  $m$  (and the same linear series); see the proof of Theorem 1.2 of [JP16]. Thus, knowing for instance the  $m = 2$  case mentioned above, we conclude that for any case  $(g, r, d)$  with  $\binom{r+2}{2} \geq 2d + 1 - g$ , the Maximal Rank Conjecture holds for all  $m$ .

We have thus far phrased the main question in terms of a restriction map for an imbedded curve in projective space, but it is instructive to rephrase the situation slightly, via the observation that if the imbedding is given by a linear series  $(\mathcal{L}, V)$  as above, then we have canonically

$$\Gamma(\mathbb{P}^r, \mathcal{O}(m)) = \text{Sym}^m V,$$

so we can rewrite (1.1) as a multiplication map on global sections:

$$(1.2) \quad \text{Sym}^m V \rightarrow \Gamma(X, \mathcal{L}^{\otimes m}).$$

In the above situation, the two maps are the same, but (1.2) makes sense even if we do not know that the linear series defines an imbedding of  $X$  into  $\mathbb{P}^r$ . Additionally, the form of (1.2) places it in the broader context of multiplication maps, which includes for instance the Petri map as well. Since our statements will be in terms of the rank of (1.2), we will freely replace  $\text{Sym}^m V$  with  $V^{\otimes m}$ .

In this paper, we use the theory of limit linear series to develop a general approach for studying ranks of multiplication maps by degenerating to a chain of smaller-genus curves. Previous approaches using limit linear series to study multiplication maps had focused on injectivity, considering a hypothetical nonzero element of the kernel, and deriving a contradiction. See for instance [EH83b] and [Tei03]. However, such an approach seems to face serious difficulties in proving sharp statements on the Maximal Rank Conjecture; see Remark 4.13. Thus, we instead study the rank of the multiplication map directly, considering collections of global sections on the reducible curve and suitable tensor products thereof. In this sense, our approach bears similarities to the tropical approach of Jensen and Payne, and their work certainly provided some evidence that such an approach is feasible. However, there are many qualitative differences in implementing the two approaches.

Although our basic degeneration machinery applies to arbitrary multiplication maps, we focus mainly on the Maximal Rank Conjecture, proving an elementary criterion (Theorem 3.9) for checking any given case of the conjecture. Our approach is relatively self-contained – the most difficult background ingredient is the fundamental smoothing theorem of Eisenbud and Harris [EH86], but there is also a substantial input from the alternative approach to limit linear series developed in [Oss06] and [Oss14]. In fact, our elementary criterion allows us to prove sharper statements than that the Maximal Rank Conjecture holds in given cases. To make precise statements, we introduce the following definition.

**Definition 1.3.** Given nonnegative  $(g, r, d, m)$  with  $m \geq 2$ ,  $r \geq 3$ , and  $g \geq (r + 1)(g + r - d)$ , we say that a smooth curve  $X$  of genus  $g$  satisfies the **maximal rank condition** for  $(r, d, m)$  if it is Brill-Noether general with respect to  $r$  and  $d$  and if there exists a linear series  $(\mathcal{L}, V)$  of rank  $r$  and degree  $d$  on  $X$  such that the multiplication map (1.2) has rank equal to  $\min \left\{ \binom{r+m}{m}, md + 1 - g \right\}$ .

With this terminology, our criterion allows us to show in a number of cases that a general chain  $X_0$  of  $g$  genus-1 curves is not in the closure of the locus in  $\mathcal{M}_g$  for which the maximal rank condition fails. In fact, our approach is quite flexible, and unlike other arguments to date involves limit linear series for curves of compact type, it potentially allows us to take into account direction of approach to the curve  $X_0$ . Thus, we can in principle prove cases of the Maximal Rank Conjecture even if it happens that all chains of  $g$  genus-1 curves are in the closure of the locus in  $\mathcal{M}_g$  where the maximal rank condition fails; see Remark 4.12. However, thus far in the examples we have studied, we have always been able to prove the stronger statement. Our main applications of our elementary criterion are to the cases  $m = 2$  and  $m = 3$ . However, we also make some simple observations on injectivity and surjectivity in some extremal cases. For simplicity, we restrict to the case  $r + g > d$ , the nonspecial case already being known. We can then summarize our results as follows.

**Theorem 1.4.** *Given  $g, r, d$  with  $r \geq 3$ ,  $r + g > d$ , and  $\rho \geq 0$ , the Maximal Rank Conjecture holds in the following circumstances:*

- (i) when  $m = 2$ ;
- (ii) when  $m = 3$ , and either  $r = 3$  with  $g \geq 7$ ,  $r = 4$  with  $g \geq 16$ , or  $r = 5$  with  $g \geq 26$ ;
- (iii) when  $g \geq (r + 1)((m + 1)^{r-1} - r)$ ;
- (iv) when  $m \geq 3$ , and either  $r + g - d = 1$  with  $2r - 3 \geq \rho + 1$ , or  $r + g - d = 2$  with  $r \geq 4$  and  $2r - 3 \geq \rho + 2$ .

Moreover, in each of the above cases, a general chain of  $g$  genus-1 curves is not in the closure of the locus of  $\mathcal{M}_g$  for which the maximal rank condition fails.

Theorem 1.4 is an immediate consequence of Theorem 6.1, Corollary 8.4, Proposition 5.7, and Corollary 7.5, with the conclusion that (iv) holds for  $m > 3$  following from the fact that surjectivity of the multiplication map for a given  $m$  implies surjectivity for all larger  $m$ .

A novel feature of our arguments is that, unlike when using degenerations to a chain of elliptic curves to prove the Brill-Noether or Gieseker-Petri theorems, our generality conditions are not solely imposed component by component, but rather in some cases involve considering multiple components at a time.

In addition, we mention that our techniques potentially apply also to studying the Strong Maximal Rank Conjecture of Aprodu and Farkas (Conjecture 5.4 of [AF11]). They are easiest to use for studying the behavior of a general linear series, so we have restricted our current presentation to that setting. However, at least in characteristic 0 (where base change and blowup can always ensure that a family of linear series specializes to a refined linear series) our techniques can potentially be used to study multiplication maps for arbitrary linear series.

Finally, we remark that in order to prove any given case of the Maximal Rank Conjecture, it is enough to produce a single smooth curve  $X$  for which the space of linear series of given rank and degree has the expected dimension  $\rho$ , and a single linear series on  $X$  such that (1.1) has the predicted rank. Indeed, while (for small  $m$  and  $d$ ) the dimension of  $\Gamma(X, \mathcal{L}^{\otimes m})$  may vary as  $X$  and  $\mathcal{L}$  vary, if we use the usual trick of twisting up by a sufficiently ample divisor on  $X$ , we can re-express the maximal rank condition in arbitrary families as a usual determinantal condition. We thus conclude that over any family of smooth curves, satisfying the maximal rank condition is an open condition in the relative moduli space of linear series. Standard dimension arguments imply that this moduli space is open over the base at any point which has fiber dimension  $\rho$ , proving that under the stated hypotheses, all nearby curves contain a nonempty open subset of linear series satisfying the maximal rank condition. For  $\rho \geq 1$ , it follows from the Gieseker-Petri and Fulton-Lazarsfeld theorems [Gie82] [FL81] that we have an open family of curves for which a dense open subset of linear series satisfies the maximal rank condition (note that the initial curve did not need to be Petri general). For  $\rho = 0$ , we instead apply the monodromy theorem of Eisenbud and Harris [EH87] to conclude that we have an open family of curves for which every linear series satisfies the maximal rank condition.

**Structure.** We begin in §2 with some calculations on doubly-marked genus-1 curves. We construct certain natural morphisms from the curves to projective spaces by varying one of the two marked points, and study when these morphisms are nondegenerate. In §3, we state our elementary criterion for the Maximal Rank Conjecture (Theorem 3.9), and give some examples of its application. In §4, we

use limit linear series to prove Theorem 3.9. In the remaining sections, we apply Theorem 3.9 to prove the various cases of Theorem 1.4: in §5 we make some observations on injectivity including a proof of case (iii); in §6 we prove the case  $m = 2$ ; in §7 we make some observations on surjectivity and prove case (iv); and finally, in §8 we prove the  $m = 3$  cases of Theorem 1.4.

**Acknowledgements.** We would like to thank Gavril Farkas for bringing to our attention the difficulties in applying the usual Eisenbud-Harris approach to multiplication maps in the situation of the Maximal Rank Conjecture, as described in Remark 4.13.

**Conventions.** The phrase “up to scalar” will always implicitly assume the scalar is nonzero.

Our arguments apply over a base field of any characteristic, although certain background results mentioned in the introduction (such as the monodromy of linear series in the  $\rho = 0$  case) may have only been proved in characteristic 0.

We do not need our base field to be algebraically closed, but nodal curves are always assumed to have all nodes and irreducible components defined over the base field.

## 2. NONDEGENERACY ON TWICE-MARKED ELLIPTIC CURVES

In this section, we study maps from elliptic curves to projective space determined by comparing values of certain tuples of sections at a point  $Q$  to a point  $P$ , as we let the point  $Q$  vary. We describe these maps explicitly, showing in the process that they are morphisms, and prove that they are nondegenerate in a family of cases of interest for the Maximal Rank Conjecture.

Given a genus-1 curve  $C$  and distinct  $P, Q$  on  $C$ , and  $c, d \geq 0$ , let  $\mathcal{L} = \mathcal{O}_C(cP + (d - c)Q)$ . Then for any  $a, b \geq 0$  with  $a + b = d - 1$ , there is a unique section (up to scaling by  $k^\times$ ) of  $\mathcal{L}$  vanishing to order at least  $a$  at  $P$  and at least  $b$  at  $Q$ . Thus, we have a uniquely determined point  $R$  such that the divisor of the aforementioned section is  $aP + bQ + R$ ; explicitly,  $R$  is determined by  $aP + bQ + R \sim cP + (d - c)Q$ , or

$$(2.1) \quad \begin{aligned} R &\sim (c - a)P + (d - c - b)Q = (c - a)P + (1 + a - c)Q \\ &= P + (a + 1 - c)(Q - P) = Q + (a - c)(Q - P). \end{aligned}$$

Thinking of  $C$  as a torsor over  $\text{Pic}^0(C)$ , we see that  $R = P$  if and only if  $Q - P$  is  $|a + 1 - c|$ -torsion, and  $R = Q$  if and only if  $Q - P$  is  $|a - c|$ -torsion. Note that (2.1) makes sense even when  $Q = P$  (in which case  $R = Q = P$ ), so we will use the formula for all  $P, Q$ , understanding that it has the initial interpretation as long as  $Q \neq P$ . To avoid trivial cases, we will assume that  $a \neq c - 1$ , and  $b \neq d - c - 1$ , or equivalently,  $a + 1 - c \neq 0$ , and  $a - c \neq 0$ .

Our general statements will follow from the case of maps to the projective line, so to simplify notation we initially restrict ourselves to this setting. We will thus study the following situation.

**Situation 2.1.** Fix  $m \geq 2$ , and set numbers  $a_1, \dots, a_m, b_1, \dots, b_m, a'_1, \dots, a'_m, b'_1, \dots, b'_m$  satisfying the above conditions (i.e.,  $a_i + b_i = d - 1$  and  $a_i - c \neq 0, -1$  for all  $i$ , and similarly for  $a'_i$  and  $b'_i$ ). Further suppose that  $\sum_i a_i = \sum_i a'_i$ .

We thus have a collection of sections  $s_i$  with divisors  $a_i P + b_i Q + R_i$ , and  $s'_i$  with divisors  $a'_i P + b'_i Q + R'_i$ . Forming their tensor product, we get sections  $s = s_1 \otimes \cdots \otimes s_m$ ,  $s' = s'_1 \otimes \cdots \otimes s'_m \in \Gamma(C, \mathcal{L}^{\otimes m})$ , with divisors

$$\left( \sum_i a_i \right) P + \left( \sum_i b_i \right) Q + R_1 + \cdots + R_m$$

and

$$\left( \sum_i a'_i \right) P + \left( \sum_i b'_i \right) Q + R'_1 + \cdots + R'_m.$$

Because we supposed that  $\sum_i a_i = \sum_i a'_i$ , we also have  $\sum_i b_i = \sum_i b'_i$ , which together imply that  $R_1 + \cdots + R_m \sim R'_1 + \cdots + R'_m$ . We thus have a rational function  $g$  on  $C$ , unique up to  $k^\times$ -scaling, such that

$$\operatorname{div} g = R_1 + \cdots + R_m - R'_1 - \cdots - R'_m.$$

Now, if  $Q - P$  is not  $|a_i - c|$ - or  $|a_i + 1 - c|$ - or  $|a'_i - c|$ - or  $|a'_i + 1 - c|$ -torsion for any  $i$ , then  $g(P)$  and  $g(Q)$  are both in  $k^\times$ , and the ratio  $g(Q)/g(P) \in k^\times$  is independent of scaling  $g$ , so is canonically determined by the choice of  $P, Q$  (as well as the discrete data we have chosen). Our immediate goal is to consider how the ratio  $g(Q)/g(P)$  varies if we fix  $P$  and let  $Q$  vary.

*Notation 2.2.* With discrete data as in Situation 2.1, suppose  $P$  is fixed. For a given  $Q \in C$ , denote by  $R_i^Q$ ,  $R_i'^Q$  and  $g^Q$  the points and rational function determined as above by  $P$  and  $Q$ . Let  $U$  be the open subset of  $C$  consisting of all  $Q$  such that  $Q - P$  is not  $|a_i - c|$ - or  $|a_i + 1 - c|$ - or  $|a'_i - c|$ - or  $|a'_i + 1 - c|$ -torsion for any  $i = 1, \dots, m$ .

Thus, for all  $Q \in U$ , we get a  $g^Q(Q)/g^Q(P) \in k^\times$ . The main technical result of this section is then the following characterization of the resulting function.

**Lemma 2.3.** *In Situation 2.1, the function  $f : U \rightarrow k^\times$  given by  $Q \mapsto g^Q(Q)/g^Q(P)$  determines a rational function on  $C$ . We then have*

$$\begin{aligned} \operatorname{div} f = & \sum_{i=1}^m ((P + \operatorname{Pic}^0(C)[|a_i - c|]) - (P + \operatorname{Pic}^0(C)[|a'_i - c|]) \\ & - (P + \operatorname{Pic}^0(C)[|a_i + 1 - c|]) + (P + \operatorname{Pic}^0(C)[|a'_i + 1 - c|])), \end{aligned}$$

where for a divisor  $D = \sum_j c_j P_j$  on  $\operatorname{Pic}^0(C)$ , the notation  $P + D$  indicates the divisor  $\sum_j c_j (P + P_j)$  on  $C$ , using the  $\operatorname{Pic}^0(C)$ -torsor structure on  $C$ .

In the above, the torsion subgroups  $\operatorname{Pic}^0(C)[n]$  should be equipped with the multiplicities arising from the inseparable degree of the appropriate multiplication map. Thus, if  $k$  has characteristic 0 or characteristic  $p$  not dividing  $n$ , then  $\operatorname{Pic}^0(C)[n]$  is a reduced divisor, but otherwise all the points of  $\operatorname{Pic}^0(C)[n]$  have coefficients given by the appropriate power of  $p$ . In particular, if the inseparable degrees are all equal to 1, then for any  $Q \in C$ , the coefficient of  $Q$  in  $\operatorname{div} f$  is expressed as

$$\#\{i : R_i^Q = Q\} - \#\{i : R_i'^Q = Q\} - \#\{i : R_i^Q = P\} + \#\{i : R_i'^Q = P\}.$$

*Proof.* Consider each  $R_i$  and each  $R'_i$  as a divisor on  $C \times C$  consisting of all points  $(Q, R_i^Q)$  (respectively,  $(Q, R_i'^Q)$ ); in fact, by (2.1) we see that each  $R_i$  and  $R'_i$  can be considered as the graph of a morphism  $C \rightarrow C$ , namely the morphism sending  $Q$  to

$P + (a+1-c)(Q-P)$  where  $a$  is  $a_i$  or  $a'_i$ . Now, set  $Z = \sum_i (P + \text{Pic}^0(C)[[a_i + 1 - c]])$  and  $Z' = \sum_i (P + \text{Pic}^0(C)[[a'_i + 1 - c]])$ .

Our first claim is that  $R_1 + \cdots + R_m + Z' \times C \sim R'_1 + \cdots + R'_m + Z \times C$  on  $C \times C$ . Noting that  $R_1 + \cdots + R_m \sim R'_1 + \cdots + R'_m$  after restriction to any fiber  $\{Q\} \times C$ , it follows that  $R_1 + \cdots + R_m - R'_1 - \cdots - R'_m \sim D \times C$  for some divisor  $D$  on  $C$ . But we now consider the restriction to  $C \times \{P\}$ , observing that by construction,

$$(2.2) \quad (R_1 + \cdots + R_m - R'_1 - \cdots - R'_m)|_{C \times \{P\}} = (Z - Z') \times \{P\}.$$

We thus conclude that we must have  $D \sim Z - Z'$ , proving our claim.

Now, let  $t, t'$  be the sections (unique up to scaling by  $k^\times$ ) of  $\mathcal{O}_{C \times C}(R_1 + \cdots + R_m + C \times Z')$  with divisors  $R_1 + \cdots + R_m + C \times Z'$  and  $R'_1 + \cdots + R'_m + C \times Z$  respectively. Our second claim is that there exist choices of  $t, t'$  such that the function  $f$  is obtained by taking the rational map  $C \times C \dashrightarrow \mathbb{P}_k^1$  induced by  $(t, t')$  and composing with the diagonal map  $U \rightarrow C \times C$ . For  $Q \in U$ , if we restrict  $(t, t')$  to  $\{Q\} \times C$ , we obtain a rational function with the same zeroes and poles as  $g^Q$ , and which is hence a valid choice for  $g^Q$ . We next observe that if we restrict  $(t, t')$  to  $C \times \{P\}$ , then by (2.2) after removing base points we have a rational function with no zeroes or poles, which is thus necessarily constant, equal to some  $z \in k^\times$ . Thus, rescaling  $t'$  by  $z$ , we may assume  $z = 1$ , which means that on each  $\{Q\} \times C$  for  $Q \in U$ , we have that  $(t, t')$  induces a choice of  $g^Q$  with  $g^Q(P) = 1$ . Thus for the given  $(t, t')$ , we have that  $g^Q(Q)/g^Q(P)$  is obtained simply by evaluation at  $(Q, Q)$ , which is the same as saying that  $f$  is induced as claimed.

It then follows that  $f$  is a rational function on  $C$ , and the desired description of its divisor likewise follows: indeed, the diagonal meets any fiber  $\{Q\} \times C$  transversely, so the last two terms in the claimed formula come directly from the restrictions of  $Z \times C$  and  $Z' \times C$ , respectively. Now, in general the diagonal may not meet the graph of the morphism  $Q \mapsto P + (a+1-c)(Q-P)$  transversely, but in any case the intersection is always identified with  $P + \text{Pic}^0(C)[a-c]$ , which thus yields the first two terms of the asserted formula for  $\text{div } f$ , as desired.  $\square$

As a sample application of Lemma 2.3, we consider when the function  $f$  is nonconstant in the case  $m = 2$ .

**Corollary 2.4.** *In the situation of Lemma 2.3, assume further that  $m = 2$  and  $C$  is not supersingular. Then the function  $f$  is nonconstant if and only if  $\{a_1, a_2\} \neq \{a'_1, a'_2\}$  and  $a_1 + a_2 \neq 2c - 1$ .*

*Proof.* By Lemma 2.3, we have that  $f$  is constant if and only if

$$\begin{aligned} 0 = D := & \sum_{i=1}^2 ((P + \text{Pic}^0(C)[[a_i - c]]) - (P + \text{Pic}^0(C)[[a'_i - c]]) \\ & - (P + \text{Pic}^0(C)[[a_i + 1 - c]]) + (P + \text{Pic}^0(C)[[a'_i + 1 - c]]). \end{aligned}$$

Without loss of generality, assume that  $a_1 \leq a_2$  and  $a'_1 \leq a'_2$ ; then because we have assumed  $a_1 + a_2 = a'_1 + a'_2$ , we have  $\{a_1, a_2\} = \{a'_1, a'_2\}$  if and only if  $a_1 = a'_1$ . Obviously, in this case, we have  $D = 0$ . Similarly, if  $a_1 + a_2 = 2c - 1$ , then also  $a'_1 + a'_2 = 2c - 1$ , so  $a_1 - c = -(a_2 + 1 - c)$ ,  $a_2 - c = -(a_1 + 1 - c)$ , and similarly for the  $a'_i$ , giving  $D = 0$  again. On the other hand, if  $a_1 \neq a'_1$ , we may assume without loss of generality that  $a_1 < a'_1$ , so that  $a_2 > a'_2$ . In particular, we have  $a_1 < a_2$ .

If  $a_1 + a_2 > 2c - 1$ , we see immediately that  $a_2 + 1 - c > c - a_1$ , but also  $a_2 + 1 - c > a_1 - c$ , so  $a_2 + 1 - c > |a_1 - c| \geq 0$ . We likewise have  $a'_2 + 1 - c > |a'_1 - c| \geq 0$ , but  $a_2 + 1 - c > a'_2 + 1 - c$ , and we easily conclude that  $|a_2 + 1 - c|$  is the (unique) maximal term appearing in the expression for  $D$ . Since  $C$  has points of order precisely  $|a_2 + 1 - c|$ , this implies that  $f$  has poles at those points, and hence is nonconstant.

Similarly, if  $a_1 + a_2 < 2c - 1$ , we see that  $|a_1 - c| = c - a_1$  is the maximal term appearing in the expression for  $D$ , implying that  $f$  has zeroes and is nonconstant.  $\square$

We now set up the more general situation of interest for us, amounting to studying morphisms to higher-dimensional projective spaces.

**Situation 2.5.** Fix  $m \geq 2$ , and  $\ell \geq 1$ , and for  $j = 0, \dots, \ell$ , set numbers  $a_1^j, \dots, a_m^j, b_1^j, \dots, b_m^j$  satisfying:

- $a_i^j + b_i^j = d - 1$  for all  $i, j$ ;
- $a_i^j - c \neq 0, -1$  for all  $i, j$ ;
- $\sum_i a_i^j$  is independent of  $j$ .

Generalizing the earlier discussion, we now have sections  $s_i^j$  with divisors  $a_i^j P + b_i^j Q + R_i^j$ , and forming tensor products yields sections  $s^j = s_1^j \otimes \dots \otimes s_m^j \in \Gamma(C, \mathcal{L}^{\otimes m})$ , with divisors

$$\left( \sum_i a_i^j \right) P + \left( \sum_i b_i^j \right) Q + R_1^j + \dots + R_m^j,$$

having the property that any two  $R_1^j + \dots + R_m^j$  are linearly equivalent. Now, if  $Q - P$  is not  $|a_i^j + 1 - c|$ -torsion for any  $i, j$ , we can normalize the  $s^j$ , uniquely up to simultaneous scalar, so that their values at  $P$  are all the same. Then provided that there is some  $j$  such that  $Q - P$  is not  $|a_i^j - c|$ -torsion for any  $i$ , considering  $(s^0(Q), \dots, s^\ell(Q))$  gives a well-defined point of  $\mathbb{P}^\ell$ .

*Notation 2.6.* With discrete data as in Situation 2.5, suppose  $P$  is fixed. For a given  $Q \in C$ , denote by  $R_i^{j,Q}$  the point determined as above by  $P$  and  $Q$ , and by  $f_Q$  the point of  $\mathbb{P}^\ell$  determined by  $(s^0(Q), \dots, s^\ell(Q))$ . Let  $U$  be the open subset of  $C$  consisting of all  $Q$  such that  $Q - P$  is not  $|a_i^j - c|$ - or  $|a_i^j + 1 - c|$ -torsion for any  $i, j$ .

Our main result is then the following.

**Corollary 2.7.** *The map  $U \rightarrow \mathbb{P}^\ell$  given by  $Q \mapsto f_Q$  extends to a morphism  $f : C \rightarrow \mathbb{P}^\ell$ .*

*If further  $C$  is not supersingular,  $m = 2$ , all the  $a_i^j$  are distinct,  $a_1^j + a_2^j \neq 2c - 1$ , and for each  $j$ , we have exactly one  $a_i^j$  less than  $c$ , then  $f$  is nondegenerate.*

*Proof.* Indeed, we can view our map as being given by  $(f_0, \dots, f_{\ell-1}, 1)$ , where  $f_j$  is the rational function constructed in Lemma 2.3 from the sections  $s^j, s^\ell$ . We thus conclude immediately that our map extends to a morphism. Moreover, nondegeneracy is equivalent to linear independence of the rational functions  $f_0, \dots, f_{\ell-1}, 1$ , whose zeroes and poles we have completely described.



Now, suppose we have the hypotheses for the nondegeneracy statement. We may also without loss of generality reorder our data so that

$$a_1^0 < a_1^1 < \cdots < a_1^\ell < c < a_2^\ell < a_2^{\ell-1} < \cdots < a_2^0.$$

Then we claim that for each  $j < \ell$ , if we set  $N_j = \max(|a_1^j + 1 - c|, |a_2^j + 1 - c|)$ , then  $f_j$  has poles at the strict  $N_j$ -torsion points of  $C$ , while none of  $f_{j+1}, \dots, f_{\ell-1}$  do. The desired linear independence follows.

For the first assertion, we have to see that the zeroes at the  $|a_i^j - c|$ -torsion and  $|a_i^j + 1 - c|$ -torsion cannot cancel the poles at the  $N_j$ -torsion. Note that  $N_j \geq a_2^j + 1 - c \geq 3$ . Certainly, we have  $|a_2^j - c| = a_2^j - c < N_j$ ,  $|a_2^j + 1 - c| = a_2^j + 1 - c < N_j$ , and  $|a_1^j + 1 - c| = c - 1 - a_1^j < c - 1 - a_1^j \leq N_j$ , so there is no problem with these. Finally, we have  $|a_1^j - c|$  relatively prime to  $|a_1^j + 1 - c|$ , so if  $N_j = |a_1^j + 1 - c|$ , the poles at the  $N_j$ -torsion cannot be cancelled by the zeroes at the  $|a_1^j - c|$ -torsion. But if  $N_j > |a_1^j + 1 - c|$ , we must have  $|a_1^j - c| - 1 = |a_1^j + 1 - c| < N_j$ , and we cannot have  $|a_1^j - c| = N_j$  because  $a_1^j + a_2^j \neq 2c - 1$ , so we must have  $|a_1^j - c| < N_j$ , and again the poles cannot be cancelled.

For the second assertion, choose  $j' > j$ ; then  $f_{j'}$  has potential poles at the  $|a_i^{j'} + 1 - c|$ -torsion and the  $|a_i^\ell - c|$ -torsion. But as above, we see that  $|a_i^{j'} + 1 - c| < N_j$  and  $|a_i^\ell - c| < N_j$  for  $i = 1, 2$ , so  $f_{j'}$  cannot have poles at the strict  $N_j$ -torsion, as desired.  $\square$

*Remark 2.8.* We see from the proof that in Corollary 2.4, we could even allow  $C$  to be supersingular as long as we assume that the base characteristic does not divide the maximum among  $|a_1 - c|, |a_2 - c|, |a_1' - c|, |a_2' - c|, |a_1 + 1 - c|, |a_2 + 1 - c|, |a_1' + 1 - c|, |a_2' + 1 - c|$ . However, for our purposes there is no harm in assuming that our curves are not supersingular.

### 3. THE ELEMENTARY CRITERION: STATEMENT AND EXAMPLES

In this section, we state a completely elementary criterion for checking that the Maximal Rank Conjecture holds in any given case. This criterion will be proved in the following section using the results of §2 and the theory of limit linear series. Although our definitions, statements, and examples are completely elementary, we do give geometric motivation for them. We have the following preliminary definitions.

**Definition 3.1.** Given  $w = (c_2, \dots, c_g)$  and  $w' = (c'_2, \dots, c'_g)$  in  $\mathbb{Z}^{g-1}$ , for  $i = 1, \dots, g$ , set

$$\epsilon_{w', w}^i = \begin{cases} 0 & : \sum_{j=i+1}^g (c'_j - c_j) > \min_{1 \leq i' \leq g} \sum_{j=i'+1}^g (c'_j - c_j) \\ 1 & : \text{otherwise.} \end{cases}$$

We say that  $(w', w)$  is **steady** if there exists  $i$  such that  $c'_j - c_j \leq 0$  for  $j \geq i$  and  $c'_j - c_j \geq 0$  for  $j < i$ .

*Remark 3.2.* The motivation for the above is that we will have a chain  $X_0$  of  $g$  genus-1 curves  $Z_i$ , with  $Q_i$  on  $Z_i$  gluing to  $P_{i+1}$  on  $Z_{i+1}$ , and with line bundles of degree  $d$  on each component. We will construct line bundles of total degree  $d$  on the chain by twisting down by  $c_i P_i$  for  $i \geq 2$  and by  $(d - c_{i+1}) Q_i$  for  $i < g$ , and gluing them all together. Given  $w, w'$  as above, if  $\mathcal{L}_w, \mathcal{L}_{w'}$  denote the line bundles resulting from this construction, then one can obtain a map  $\mathcal{L}_{w'} \rightarrow \mathcal{L}_w$  which is injective on some components and vanishes identically on the other components,

and  $\epsilon_{w',w}^i = 1$  exactly when the map is injective on  $Z_i$ ; see Proposition 4.6 below. In some cases, the definition of  $\epsilon_{w,w'}^i$  can be simplified substantially; see Remark 3.14 below.

**Definition 3.3.** Given  $g, r, d > 0$  with  $g \geq (r+1)(r+g-d)$ , a  $(g, r, d)$ -**sequence**  $\delta_1, \dots, \delta_g$  is a sequence of  $g$  integers between 0 and  $r$ , with each integer between 0 and  $r$  occurring at least  $r+g-d$  times, and satisfying the condition that for each  $i = 1, \dots, g$ , no integer strictly less than  $\delta_i$  occurs among  $\delta_1, \dots, \delta_i$  strictly fewer times than  $\delta_i$  does.

More generally, given also  $a \geq 0$ , an  $a$ -**shifted**  $(g, r, d)$ -**sequence**  $\delta_1, \dots, \delta_g$  is a  $(g, r, d)$ -sequence in which every integer between 0 and  $r$  occurs at least  $a+r+g-d$  times.

Although in most cases there are many possible  $(g, r, d)$  sequences, we will typically work with sequences which are either nondecreasing, or which simply cycle through the numbers  $0, \dots, r$  up to a total length of  $g$ .

We use  $(g, r, d)$ -sequences to index particular families of limit linear series. The relevant numerical construction is the following.

**Definition 3.4.** Given a  $(g, r, d)$ -sequence  $\vec{\delta} = \delta_1, \dots, \delta_g$ , and  $m \geq 2$ , we define tables  $T'(\vec{\delta})$  and  $T(\vec{\delta})$  as follows:

First construct the  $(r+1) \times g$  table  $T'(\vec{\delta})$  with  $(j+1)$ st row consisting of  $(a_j^1, b_j^1), (a_j^2, b_j^2), \dots, (a_j^g, b_j^g)$ , where  $a_j^i = j$ , for each  $i = 1, \dots, g$  we have  $b_j^i = d - a_j^i$  if  $j = \delta_i$  and  $b_j^i = d - 1 - a_j^i$  if  $j \neq \delta_i$ , and for each  $i = 1, \dots, g-1$  we have  $a_j^{i+1} = d - b_j^i$ .

Then we construct the  $\binom{r+m}{m} \times g$  table  $T(\vec{\delta})$ , with rows indexed by  $\vec{j} = (j_1, \dots, j_m)$  (with  $0 \leq j_1 \leq j_2 \leq \dots \leq j_m \leq r$ ), and each entry being a pair of integers  $(a_{\vec{j}}^i, b_{\vec{j}}^i)$ , by setting the  $(j_1, \dots, j_m)$ th row of  $T(\vec{\delta})$  to be the sum of the  $j_n$ th rows of  $T'(\vec{\delta})$ , for  $n = 1, \dots, m$ .

More generally, given an  $a$ -shifted  $(g, r, d)$ -sequence, define the tables  $T'(\vec{\delta})$  and  $T(\vec{\delta})$  just as above, except that we start with  $a_j^1 = a + j$  for  $j = 0, \dots, r$ .

Thus, in the  $a$ -shifted case, all the  $a_j^i$  are  $a$  larger than in the usual case, and all the  $a_j^i$  are  $ma$  larger. This is convenient for certain reduction arguments.

*Remark 3.5.* The geometric interpretation of the above is that the  $a_j^i$  and  $b_j^i$  keep track of orders of vanishing of sections at  $P_i$  and  $Q_i$  respectively. Each column of  $T'(\vec{\delta})$  describes a  $\mathfrak{g}_d^r$  on  $Z_i$  in terms of the orders of vanishing of its sections, and taken together these form a limit linear series on  $X_0$ ; see the proof of Theorem 3.9 in §4 below. Then each row of  $T'(\vec{\delta})$  can be used to construct a global section of a line bundle in a multidegree which is determined by the  $a_j^i$  and  $b_j^i$ , and the rows of  $T(\vec{\delta})$  correspond to  $m$ -fold tensor products of these global sections.

Given also a  $(g-1)$ -tuple of integers  $w$ , we will generate a second table  $T_w(\vec{\delta})$  from  $T(\vec{\delta})$ , of the same dimensions, but where we allow blank entries whose distribution is determined by  $w$ . We also extend the notion of steadiness to this situation.

**Definition 3.6.** In the situation of Definition 3.4, suppose that we are also given a  $w = (c_2, \dots, c_g) \in \mathbb{Z}^{g-1}$ . Then define the table  $T_w(\vec{\delta})$  obtained from  $T(\vec{\delta})$  by erasing certain entries as follows: for the row of  $T(\vec{\delta})$  indexed by  $\vec{j} = (j_1, \dots, j_m)$ ,

let  $w' = (a_{\vec{j}}^2, \dots, a_{\vec{j}}^g)$ . Then for  $i = 1, \dots, g$ , the  $i$ th entry in the  $\vec{j}$ th row of  $T(\vec{\delta})$  is erased in  $T_w(\vec{\delta})$  if the  $\epsilon_{w', w}^i$  of Definition 3.1 is equal to 0.

In addition, we say that  $T(\vec{\delta})$  is **steady** with respect to  $w$  if for each  $\vec{j}$ , setting  $w' = (a_{\vec{j}}^2, \dots, a_{\vec{j}}^g)$  as above, we have that  $(w', w)$  is steady.

**Definition 3.7.** In the situation of Definition 3.6, we say that  $T_w(\vec{\delta})$  is  **$N$ -expungeable** if there exists a choice of some  $N$  rows of  $T_w(\vec{\delta})$  such that the following iterative row-dropping procedure can be used to eliminate all rows of  $T_w(\vec{\delta})$ :

- (i) first, drop the  $\binom{r+m}{m} - N$  rows of  $T_w(\vec{\delta})$  which are not the chosen  $N$  rows;
- (ii) if for some  $i = 1, \dots, g$ , there is a remaining row  $\vec{j}$  whose  $i$ th entry is not erased in  $T_w(\vec{\delta})$ , and such that  $a_{\vec{j}}^i$  is strictly minimal among all remaining rows whose  $i$ th entry is not erased in  $T_w(\vec{\delta})$ , then the  $\vec{j}$  row can be dropped;
- (iii) if for some  $i = 1, \dots, g$ , there is a remaining row  $\vec{j}$  whose  $i$ th entry is not erased in  $T_w(\vec{\delta})$ , and such that  $b_i^{\vec{j}}$  is strictly minimal among all remaining rows whose  $i$ th entry is not erased in  $T_w(\vec{\delta})$ , then the  $\vec{j}$  row can be dropped;
- (iv) if for some  $i = 1, \dots, g$ , there are at most two remaining rows whose  $i$ th entries are not erased in  $T_w(\vec{\delta})$ , then these rows can be dropped;
- (v) if for some  $i = 1, \dots, g$ ,  $j = 0, \dots, r$  with  $j \neq \delta_i$ , there is some  $n \geq 0$  such that for every remaining row  $\vec{j} = (j_1, \dots, j_m)$  whose  $i$ th entry is not erased in  $T_w(\vec{\delta})$ , at least  $n$  of the  $j_\ell$  are equal to  $j$ , and there is a unique row  $\vec{j} = (j_1, \dots, j_m)$  for which exactly  $n$  of the  $j_\ell$  are equal to  $j$ , then the  $\vec{j}$  row can be dropped;
- (vi) if for some  $i = 1, \dots, g - 1$  we have remaining rows  $\vec{j}^1 = (j_1^1, \dots, j_m^1)$ ,  $\vec{j}^2 = (j_1^2, \dots, j_m^2)$  satisfying the conditions below, then the rows  $\vec{j}^1, \vec{j}^2$  can be dropped:
  - $a_{\vec{j}^1}^i = a_{\vec{j}^2}^i < a_{\vec{j}'}^i$  for every remaining row  $\vec{j}' \neq \vec{j}^1, \vec{j}^2$  in column  $i$ ;
  - $b_{\vec{j}^1}^{i+1} = b_{\vec{j}^2}^{i+1} < b_{\vec{j}'}^{i+1}$  for every remaining row  $\vec{j}' \neq \vec{j}^1, \vec{j}^2$  in column  $i + 1$ ;
  - for at least one of  $i' = i$  or  $i' = i + 1$ , we have all but exactly two of the  $j_1^1, \dots, j_m^1$  are equal to  $\delta_{i'}$ , all but exactly two of the  $j_1^2, \dots, j_m^2$  are equal to  $\delta_{i'}$ , and  $a_{\vec{j}^1}^{i'} \neq a_{(\delta_{i'}, \dots, \delta_{i'})}^{i'} - 1$ .
- (vii) if  $m = 2$  and for some  $i = 1, \dots, g$  and  $n \geq 2$  we have remaining rows  $\vec{j}^e = (j_1^e, j_2^e)$  for  $e = 1, \dots, n$  satisfying the conditions below, then the rows  $\vec{j}^1, \dots, \vec{j}^n, (\delta_i, \delta_i)$  can be dropped:
  - $\delta_{i'} = \delta_i$  for  $i' = i, \dots, i + n - 1$ ;
  - for  $e = 1, \dots, n$ , we have  $j_1^e < \delta_i < j_2^e$ ;
  - the value of  $a_{\vec{j}^e}^i$  is independent of  $e \in \{1, \dots, n\}$ ;
  - for  $i' = i, \dots, i + n - 1$ , all rows  $\vec{j}^1, \dots, \vec{j}^n$  appear in the  $i'$ th column, and the only other remaining row  $\vec{j}$  which may appear in these columns is the  $(\delta_i, \delta_i)$  row.

Note that the one-row portion of Rule (iv) follows from either Rule (ii) or Rule (iii), while the two-row portion of Rule (iv) is a special case of Rule (v); we list it separately because it is so much simpler, and in practice is used more frequently than the general case.

*Remark 3.8.* The geometric idea behind the construction of  $T_w(\vec{\delta})$  from  $T(\vec{\delta})$  is that we take each row of  $T(\vec{\delta})$ , corresponding to a global section in a certain multidegree, and consider its image in the multidegree determined by  $w$ . The erased columns correspond to the components on which the map between multidegrees vanishes identically. Thus, while the rows of  $T(\vec{\delta})$  correspond to a collection of global sections in different multidegrees, the rows of  $T_w(\vec{\delta})$  correspond to a collection of global sections which all lie in the multidegree determined by  $w$ .

The idea behind the definition of  $N$ -expungeable is then that we can find a subset of  $N$  rows for which we can prove that the corresponding global sections are linearly independent. Each rule (other than Rule (i)) allows us to drop rows when we can see that the corresponding global sections cannot occur with nonzero coefficient in a linear dependence. For instance, in Rule (ii) we have that the order of vanishing at  $P_i$  of the global section in question is strictly smaller than the orders of vanishing of all the other remaining sections, so this section cannot have nonzero coefficient in a linear dependence. Rule (iii) is the same with  $Q_i$  in place of  $P_i$ . Rule (v) is again the same, except that it involves looking at the order of vanishing at a point other than  $P_i$  or  $Q_i$ . Rules (vi) and (vii) are the most interesting. Using the geometric arguments from §2, they describe circumstances under which a collection of sections may be linearly dependent on each component separately, but where, provided the  $P_i$  and  $Q_i$  are general (considering multiple components at once), there cannot be a linear dependence which holds on all of the components simultaneously.

**Theorem 3.9.** *Given  $m \geq 2$ ,  $r \geq 3$ , and positive  $g, d$  with  $g \geq (r+1)(r+g-d)$ , suppose there exists a  $(g, r, d)$ -sequence  $\vec{\delta} = (\delta_1, \dots, \delta_g)$  and a  $w = (c_1, \dots, c_{g-1}) \in \mathbb{Z}^{g-1}$  such that  $T_w(\vec{\delta})$  is  $N$ -expungeable.*

*Then there exists a Brill-Noether-general smooth curve  $X$  and a  $\mathfrak{g}_d^r(\mathcal{L}, V)$  on  $X$  such that the resulting  $m$ -multiplication map has rank at least  $N$ , and more specifically, there exists a chain  $X_0$  of genus-1 curves such that any regular one-parameter family of smooth curves approaching  $X_0$  contains some  $X$  as above. If further  $T(\vec{\delta})$  is steady with respect to  $w$ , then  $X_0$  is not in the closure of the locus consisting of smooth curves  $X$  for which all  $\mathfrak{g}_d^r$ s on  $X$  have  $m$ -multiplication map of rank strictly smaller than  $N$ .*

*In particular, if  $N = \min\left(\binom{r+m}{m}, md+1-g\right)$ , the Maximal Rank Conjecture holds for  $(g, r, d, m)$ , and under the additional steadiness hypothesis, the locus in  $\overline{\mathcal{M}}_g$  consisting of chains of genus-1 curves is not in the closure of the locus of  $\mathcal{M}_g$  for which the maximal rank condition fails.*

*Remark 3.10.* In fact, the rules listed in Definition 3.7 are not the most general which we could include for Theorem 3.9. For instance, the results of §2 could give more general forms of Rules (vi) and (vii), and one can also prove more general versions of rules similar to Rule (v). For the sake of avoiding unnecessary complexity, we have chosen to only include rules which we will actually use to prove cases of the Maximal Rank Conjecture. However, we have formulated our proof of Theorem 3.9 using Corollary 4.11 below so that additional rules may easily be added as they may become necessary in further cases.

Before proceeding to the proof of the theorem, we give several examples of its application. The first two are very simple cases for  $r = 3$ ,  $m = 2$ , but as we will see in the proof of Theorem 6.1 below, these examples fully handle the case  $r = 3$  and  $m = 2$ , and also constitute the base for the general case with  $m = 2$ .

**Example 3.11.** Consider the  $r = 3$ ,  $m = 2$  canonical case, so that  $g = 4$  and  $d = 6$ . In this case, the only possible  $(g, r, d)$ -sequence is  $\vec{\delta} = 0, 1, 2, 3$ , which gives  $T'(\vec{\delta})$  as follows.

0	6	0	5	1	4	2	3
1	4	2	4	2	3	3	2
2	3	3	2	4	2	4	1
3	2	4	1	5	0	6	0

We then get  $T(\vec{\delta})$  as follows.

		10	2	6	6	4	8	
(0,0)	0	12	0	10	2	8	4	6
(0,1)	1	10	2	9	3	7	5	5
(0,2)	2	9	3	7	5	6	6	4
(1,1)	2	8	4	8	4	6	6	4
(0,3)	3	8	4	6	6	4	8	3
(1,2)	3	7	5	6	6	5	7	3
(1,3)	4	6	6	5	7	3	9	2
(2,2)	4	6	6	4	8	4	8	2
(2,3)	5	5	7	3	9	2	10	1
(3,3)	6	4	8	2	10	0	12	0
		10	2	6	6	4	8	

The highlighted entries in the above table are precisely the non-erased entries in  $T_w(\vec{\delta})$ , if we choose  $w = (2, 6, 8)$ ; we have placed the  $c_i$  and  $md - c_i$  at the top and bottom of the table in order to make the erasure procedure clearer.

Since  $\binom{r+2}{2} = 10 > 2d + 1 - g = 9$ , this is a surjective case, and to prove surjectivity in this case we may drop any one row using Rule (i). If for instance we drop the  $(0, 3)$  row, we see that there are no remaining repetitions among the  $a_j^i$  in any column, so we have that  $T_w(\vec{\delta})$  is 9-expungeable simply by repeated application of Rule (ii), proving the desired surjectivity.

**Example 3.12.** Next consider the case  $r = 3$ ,  $m = 2$ ,  $g = 5$  and  $d = 7$ , and choose the  $(g, r, d)$ -sequence  $\vec{\delta} = 0, 1, 2, 3, 0$ . which gives  $T'(\vec{\delta})$  as follows.

0	7	0	6	1	5	2	4	3	4
1	5	2	5	2	4	3	3	4	2
2	4	3	3	4	3	4	2	5	1
3	3	4	2	5	1	6	1	6	0

We then get  $T(\vec{\delta})$  as follows.

		12	2	8	6	6	8	4	10	
(0,0)	0	14	0	12	2	10	4	8	6	8
(0,1)	1	12	2	11	3	9	5	7	7	6
(0,2)	2	11	3	9	5	8	6	6	8	5
(1,1)	2	10	4	10	4	8	6	6	8	4
(0,3)	3	10	4	8	6	6	8	5	9	4
(1,2)	3	9	5	8	6	7	7	5	9	3
(1,3)	4	8	6	7	7	5	9	4	10	2
(2,2)	4	8	6	6	8	6	8	4	10	2
(2,3)	5	7	7	5	9	4	10	3	11	1
(3,3)	6	6	8	4	10	2	12	2	12	0
		12	2	8	6	6	8	4	10	

The highlighted entries above correspond to  $T_w(\vec{\delta})$  if we choose  $w = (2, 6, 8, 10)$ .

This case is both surjective and injective, so we cannot drop any rows from Rule (i). However, we can drop the last four rows by applying Rule (iii) twice and Rule (iv) once to the last column, and then apply Rule (iv) to the third column to drop the (0, 3) and (1, 2) rows. After this, no repetitions remain among either the  $a_j^i$  or  $b_j^i$  in any column, so we can drop the rest of the rows using either Rule (ii) or Rule (iii).

The following example is the first requiring the use of Rule (vi) or (vii), and is the first of the sequence of ‘critical’ cases for  $m = 2$ , treated more generally in Proposition 6.3 below.

**Example 3.13.** Consider the case  $m = 2$ ,  $r = 4$ ,  $g = 10$ , and  $d = 12$ , and take the  $(g, r, d)$ -sequence  $\vec{\delta} = 0, 0, 1, 1, 2, 2, 3, 3, 4, 4$ . This gives  $T'(\vec{\delta})$  as follows.

0	12	0	12	0	11	1	10	2	9	3	8	4	7	5	6	6	5	7	4
1	10	2	9	3	9	3	9	3	8	4	7	5	6	6	5	7	4	8	3
2	9	3	8	4	7	5	6	6	6	6	6	6	5	7	4	8	3	9	2
3	8	4	7	5	6	6	5	7	4	8	3	9	3	9	3	9	2	10	1
4	7	5	6	6	5	7	4	8	3	9	2	10	1	11	0	12	0	12	0

We then get  $T(\vec{\delta})$  as follows.

	22	2	20	4	17	7	15	9	12	12	9	15	7	17	4	20	2	22		
(0,0)	0	24	0	24	0	22	2	20	4	18	6	16	8	14	10	12	12	10	14	8
(0,1)	1	22	2	21	3	20	4	19	5	17	7	15	9	13	11	11	13	9	15	7
(0,2)	2	21	3	20	4	18	6	16	8	15	9	14	10	12	12	10	14	8	16	6
(1,1)	2	20	4	18	6	18	6	18	6	16	8	14	10	12	12	10	14	8	16	6
(0,3)	3	20	4	19	5	17	7	15	9	13	11	11	13	10	14	9	15	7	17	5
(1,2)	3	19	5	17	7	16	8	15	9	14	10	13	11	11	13	9	15	7	17	5
(0,4)	4	19	5	18	6	16	8	14	10	12	12	10	14	8	16	6	18	5	19	4
(1,3)	4	18	6	16	8	15	9	14	10	12	12	10	14	9	15	8	16	6	18	4
(2,2)	4	18	6	16	8	14	10	12	12	12	12	12	12	10	14	8	16	6	18	4
(1,4)	5	17	7	15	9	14	10	13	11	11	13	9	15	7	17	5	19	4	20	3
(2,3)	5	17	7	15	9	13	11	11	13	10	14	9	15	8	16	7	17	5	19	3
(2,4)	6	16	8	14	10	12	12	10	14	9	15	8	16	6	18	4	20	3	21	2
(3,3)	6	16	8	14	10	12	12	10	14	8	16	6	18	6	18	6	18	4	20	2
(3,4)	7	15	9	13	11	11	13	9	15	7	17	5	19	4	20	3	21	2	22	1
(4,4)	8	14	10	12	12	10	14	8	16	6	18	4	20	2	22	0	24	0	24	0
	22	2	20	4	17	7	15	9	12	12	9	15	7	17	4	20	2	22		

The highlighted entries above correspond to the  $T_w(\vec{\delta})$  which we obtain if we choose  $w = (2, 4, 7, 9, 12, 15, 17, 20, 22)$ . If we go from left to right we can use Rule

(ii) to drop rows  $(0, 0)$  and  $(0, 1)$  in the first column,  $(0, 2)$  in the second column,  $(0, 3)$  and  $(1, 1)$  in the third column, and  $(1, 2)$  in the fourth column. Then, using Rule (iii) we can drop rows  $(4, 4)$  and  $(3, 4)$  from the last column, row  $(2, 4)$  from the ninth column, rows  $(1, 4)$  and  $(3, 3)$  from the eighth column, and row  $(2, 3)$  from the seventh column. This leaves only rows  $(0, 4)$ ,  $(1, 3)$  and  $(2, 2)$  in the fifth and sixth columns, which can be dropped using either Rule (vi) (together with Rule (iv)) or Rule (vii).

Rule (v) is not used until  $m \geq 3$ , but see Example 8.3 for a case in which it is used repeatedly and crucially.

*Remark 3.14.* In the situation of Definition 3.1, set

$$M = \min_{1 \leq i \leq g} \sum_{j=i+1}^g (c'_j - c_j).$$

Given our geometric motivation for Definition 3.1, we observe that if for some  $i$  we have  $c'_i < c_i$ , then the map from the multidegree determined by  $w'$  to the multidegree determined by  $w$  should vanish identically on  $Z_i$ , since we are twisting down more at  $P_i$  in the latter. And indeed, in this case we have

$$\sum_{j=i+1}^g (c'_j - c_j) > \sum_{j=i}^g (c'_j - c_j) \geq M,$$

so  $\epsilon_{w',w}^i = 0$ , as expected. Similarly, if  $d - c'_i < d - c_i$ , considering twists by  $Q_{i-1}$  we should have vanishing on  $Z_{i-1}$ , and we see that since  $c_i < c'_i$ , we have

$$\sum_{j=i}^g (c'_j - c_j) > \sum_{j=i+1}^g (c'_j - c_j) \geq M,$$

so  $\epsilon_{w',w}^{i-1} = 0$ , again as expected. We conclude:

If  $c'_i < c_i$  or  $d - c'_{i+1} < d - c_{i+1}$ , then necessarily  $\epsilon_{w',w}^i = 0$ .

The converse doesn't hold in general, but it does hold when the signs of  $c'_i - c_i$  are weakly decreasing, so that there is never a 0 before a positive number or a negative number before a nonnegative number. In this situation, if  $c'_i - c_i$  is never 0, the minimum  $M$  occurs at the unique  $i$  such that  $c'_i > c_i$  and  $c'_{i+1} < c_{i+1}$  (or at  $i = 1$  if  $c'_{i+1} < c_{i+1}$  for all  $i$ , and at  $i = g$  if  $c'_i > c_i$  for all  $i$ ). If  $c'_i - c_i = 0$  for some  $i$ , the minimum  $M$  occurs for the  $i$  such that that  $c'_i - c_i = 0$  or  $c'_{i+1} - c_{i+1} = 0$ . In both cases, these are precisely the  $i$  such that  $c'_i \geq c_i$  and  $d - c'_{i+1} \geq d - c_{i+1}$ . This situation will hold in particular when  $w$  is 'unimaginative' – see Definition 5.3 below.

Thus, in the above situation we have  $\epsilon_{w',w}^i = 1$  if and only if  $c'_i \geq c_i$  and  $d - c'_{i+1} \geq d - c_{i+1}$  (where we ignore the appropriate inequality if  $i = 1$  or  $i = g$ ). We also have that the  $i$  for which  $\epsilon_{w',w}^i = 1$  are contiguous.

#### 4. PROOF OF THE ELEMENTARY CRITERION

In order to prove Theorem 3.9, we begin by discussing generalities on the behavior of multiplication maps under degenerations, and the relationship to limit linear series. Although we will ultimately consider the case of powers of a single line bundle, the arguments are the same for the more general situation of products of distinct line bundles, and we consequently present them in that generality. This

is in any case arguably clearer because ultimately even when considering powers of a single line bundle, we will want to consider distinct extensions to the reducible special fiber.

We begin by reviewing the theory of limit linear series for curves of compact type. We first treat the situation on a single reducible curve, and then discuss what happens in degenerating families.

**Definition 4.1.** Let  $X_0$  be a curve of compact type, with dual graph  $\Gamma$ , and components  $Z_v$  for  $v \in V(\Gamma)$ . Then a **limit linear series** of rank  $r$  and degree  $d$  on  $X_0$  is a tuple  $(\mathcal{L}^v, V^v)_{v \in V(\Gamma)}$  of linear series of rank  $r$  and degree  $d$  on the components  $Z_v$  of  $X_0$  satisfying the following condition: for any  $e \in E(\Gamma)$ , let  $v, v'$  be the adjacent vertices, so that  $Z_v$  and  $Z_{v'}$  intersect in a node  $P_e$  of  $X_0$ , and let  $a_0^{(e,v)} < \dots < a_r^{(e,v)}$  and  $a_0^{(e,v')} < \dots < a_r^{(e,v')}$  be the vanishing sequences of  $(\mathcal{L}^v, V^v)$  and  $(\mathcal{L}^{v'}, V^{v'})$  at  $P_e$ , respectively. Then we require that

$$a_j^{(e,v)} + a_{r-j}^{(e,v')} \geq d \quad \text{for } j = 0, \dots, r.$$

We say that a limit linear series is **refined** if the above inequality is an equality for all  $e$  and  $j$ .

Now we recall some ideas from [Oss14] which will play an important role for us. First, we want to consider all possible multidegrees, and construct maps between them. This will require making some choices as follows:

**Situation 4.2.** Let  $X_0$  be a curve of compact type with dual graph  $\Gamma$  and components  $Z_v$  for  $v \in V(\Gamma)$ . For each  $Z_v$ , if  $Z'$  is the closure of  $X_0 \setminus Z_v$ , let  $\mathcal{O}^v$  be the line bundle on  $X_0$  which is  $\mathcal{O}_{Z_v}(-(Z' \cap Z_v))$  on  $Z_v$  and  $\mathcal{O}_{Z'}(Z_v \cap Z')$  on  $Z'$ . For each  $Z_v$ , also choose a section  $\sigma_v \in \Gamma(X_0, \mathcal{O}^v)$  which vanishes precisely on  $Z_v$ . Also choose an isomorphism  $\theta : \bigotimes_{v \in V(\Gamma)} \mathcal{O}^v \xrightarrow{\sim} \mathcal{O}_{X_0}$ .

Note that each  $\mathcal{O}^v$  is unique up to isomorphism because  $X_0$  is of compact type, but in general  $\sigma_v$  is not unique up to scaling: indeed, if  $X_0 \setminus Z_v$  is disconnected, then  $\sigma_v$  may be scaled independently on each connected component. This is potentially useful; see Remark 4.12.

From (the line bundles underlying) a limit linear series we then construct a collection of line bundles in different multidegrees, together with maps between them, as follows. Choose a ‘base component’  $Z_{v_0}$  of  $X_0$ , and if  $\omega_0$  is the multidegree assigning degree  $d$  to  $Z_{v_0}$  and degree 0 to every other component of  $X_0$ , fix a choice of  $\mathcal{L}_{\omega_0}$  as the line bundle obtained by gluing the line bundles  $\mathcal{L}^{v_0}$  on  $Z_{v_0}$  and  $\mathcal{L}^v(-dP_{v,v_0})$  on  $Z_v$  for  $v \neq v_0$ , where  $P_{v,v_0}$  is the node on  $Z_v$  in the direction of  $Z_{v_0}$ .

Then given an arbitrary  $\omega$ , there is a unique collection of nonnegative integers  $a_v$  for  $v \in V(\Gamma)$  such that at least one  $a_v$  is equal to 0, and such that  $\bigotimes_{v \in V(\Gamma)} (\mathcal{O}^v)^{\otimes a_v}$  has multidegree  $\omega - \omega_0$ . Then set

$$\mathcal{L}_\omega = \mathcal{L}_{\omega_0} \otimes \left( \bigotimes_v (\mathcal{O}^v)^{\otimes a_v} \right).$$

We then also have a morphism  $\mathcal{L}_{\omega_0} \rightarrow \mathcal{L}_\omega$  induced by the appropriate tensor product of the sections  $\sigma_v$ .

More generally, given another multidegree  $\omega'$ , if  $\bigotimes_{v \in V(\Gamma)} (\mathcal{O}^v)^{\otimes a'_v}$  has multidegree  $\omega' - \omega_0$ , we get a morphism  $\mathcal{L}_{\omega'} \rightarrow \mathcal{L}_\omega$  as follows: let  $b = \max_v (a'_v - a_v)$ , and



for each  $Z_v$ , set  $c_v = a_v - a'_v + b$ . Then we again have all  $c_v$  nonnegative with at least one equal to 0, and

$$\mathcal{L}_\omega \cong \mathcal{L}_{\omega'} \otimes \left( \bigotimes_v (\mathcal{O}^v)^{\otimes c_v} \right).$$

More precisely, note that since we cannot have  $a_v > 0$  for all  $v$ , we have  $b \geq 0$ , and thus according to our definitions we have

$$\mathcal{L}_\omega \otimes \left( \bigotimes_v (\mathcal{O}^v)^b \right) = \mathcal{L}_{\omega'} \otimes \left( \bigotimes_v (\mathcal{O}^v)^{\otimes c_v} \right),$$

so we obtain an induced morphism  $\mathcal{L}_{\omega'} \rightarrow \mathcal{L}_\omega$  from the appropriate tensor product of the  $\sigma_v$ , together with  $\theta^{\otimes b}$ . This morphism vanishes precisely on the components  $Z_v$  of  $X_0$  for which  $c_v > 0$ .

Finally, we note that we have restriction maps as follows: given a component  $Z_v$ , let  $\omega_v$  be the multidegree having degree  $d$  on  $Z_v$  and degree 0 on all other components. Then for any multidegree  $\omega$ , we obtain a morphism  $\mathcal{L}_\omega \rightarrow \mathcal{L}^v$ , unique up to scalar, by composing our constructed morphism  $\mathcal{L}_\omega \rightarrow \mathcal{L}_{\omega_v}$  with the restriction map  $\mathcal{L}_{\omega_v}|_{Z_v} \xrightarrow{\sim} \mathcal{L}^v$ . Depending on the choice of  $\omega$ , this restriction map may vanish uniformly, but this will not happen in most cases of interest; see Proposition 4.6 below.

Next, we specialize to the case that  $X_0$  is a chain of curves.

**Situation 4.3.** In Situation 4.2, suppose further that  $X_0 = Z_1 \cup \dots \cup Z_n$  is a chain of curves, obtained by gluing points  $Q_i$  on  $Z_i$  to  $P_{i+1}$  on  $Z_{i+1}$  for  $i = 1, \dots, n-1$ .

In this setting, we modify the above notation to use indices  $i = 1, \dots, n$  in place of  $v \in V(\Gamma)$ , except that we will not use  $\omega_i$  to denote the multidegree concentrated on the  $i$ th components. We allow  $n > g$  because we will need to consider the case that some components are rational. We then make the following definition:

**Definition 4.4.** A limit linear series  $(\mathcal{L}^i, V^i)$  on  $X_0$  is **chain-adaptable** if, for  $i = 2, \dots, n-1$ , there exist sections  $s_0^i, \dots, s_r^i$  in  $V^i$  such that  $\text{ord}_{P_i} s_0^i < \text{ord}_{P_i} s_1^i < \dots < \text{ord}_{P_i} s_r^i$  recovers the vanishing sequence of  $V^i$  at  $P_i$ , and  $\text{ord}_{Q_i} s_r^i < \text{ord}_{Q_i} s_{r-1}^i < \dots < \text{ord}_{Q_i} s_0^i$  recovers the vanishing sequence of  $V^i$  at  $Q_i$ .

Note that in the definition above, the imposed strict inequalities automatically imply that the vanishing sequences are recovered from the  $s_j^i$ . In this situation, it is often useful to consider an alternative encoding of multidegrees as follows:

*Notation 4.5.* Given a tuple  $w = (c_2, \dots, c_g)$  of integers, if we also specify a total degree  $d'$  (which will be equal to  $d$  or  $md$  in our situation), we obtain a unique multidegree  $\text{md}_{d'}(w)$  by setting the degree to  $c_2$  on  $Z_1$ , to  $c_{i+1} - c_i$  on  $Z_i$  for  $1 < i < g$ , and to  $d' - c_g$  on  $Z_g$ .

The significance of the  $c_i$  is that if  $\mathcal{L}^i$  are the line bundles on each  $Z_i$  having degree  $d'$ , we obtain the line bundle  $\mathcal{L}_{\text{md}(w)}$  by gluing together the following:

- $\mathcal{L}^1(-(d' - c_2)Q_1)$  on  $Z_1$ ;
- $\mathcal{L}^i(-c_i P_i - (d' - c_{i+1})Q_i)$  on  $Z_i$  for  $1 < i < g$ ;
- and  $\mathcal{L}^g(-c_g P_g)$  on  $Z_g$ .

We write  $\mathcal{L}_{\text{md}(w)}$  in place of  $\mathcal{L}_{\text{md}_{d'}(w)}$  because in this case the  $d'$  is already determined as  $\deg \mathcal{L}$ .

We describe the maps between different multidegrees as follows.

**Proposition 4.6.** *Given  $w = (c_2, \dots, c_g)$  and  $w' = (c'_2, \dots, c'_g)$  in  $\mathbb{Z}^{g-1}$ , for any choice of line bundle  $\mathcal{L}_{\omega_0}$  the natural map  $\mathcal{L}_{\text{md}(w')} \rightarrow \mathcal{L}_{\text{md}(w)}$  vanishes on a given  $Z_i$  if and only if  $\epsilon_{w',w}^i = 0$ .*

*In particular, as long as  $0 \leq c'_i \leq d'$  for  $i = 2, \dots, g$  none of the restriction maps  $\mathcal{L}_{\text{md}(w')} \rightarrow \mathcal{L}^i$  vanish uniformly.*

*Proof.* Observe that for any  $i \leq g-1$ , the multidegree of  $\mathcal{O}^{1,i} := \bigotimes_{i'=1}^i \mathcal{O}^{i'}$  is zero on all components except  $Z_i$  and  $Z_{i+1}$ ; it is  $-1$  on  $Z_i$  and  $1$  on  $Z_{i+1}$ . In the notation introduced above, it is  $\text{md}_0(w'')$  where  $w'' = (c''_2, \dots, c''_g)$  with all  $c''_{i'}$  equal to  $0$  except that  $c''_{i+1} = -1$ . We then see that if we want to go from  $\mathcal{L}_{\text{md}(w')}$  to  $\mathcal{L}_{\text{md}(w)}$ , we can first tensor by  $(\mathcal{O}^{1,g-1})^{\otimes c'_g - c_g}$  to get the desired degree on  $Z_g$ , then by  $(\mathcal{O}^{1,g-2})^{\otimes c'_{g-1} - c_{g-1}}$  to get the desired degree on  $Z_{g-1}$ , and so forth. Thus, we conclude that

$$\mathcal{L}_{\text{md}(w)} \cong \mathcal{L}_{\text{md}(w')} \bigotimes_{i=1}^{g-1} (\mathcal{O}^{1,i})^{\otimes c'_{i+1} - c_{i+1}} = \mathcal{L}_{\text{md}(w')} \bigotimes_{i=1}^{g-1} (\mathcal{O}^i)^{\otimes \sum_{j=i+1}^g c'_j - c_j}.$$

If we set  $M = \min_{1 \leq i \leq g} \sum_{j=i+1}^g c'_j - c_j$ , then we have  $M \leq 0$  by considering  $i = g$ , and we can write

$$\mathcal{L}_{\text{md}(w)} \cong \mathcal{L}_{\text{md}(w')} \bigotimes_{i=1}^g (\mathcal{O}^i)^{\otimes (\sum_{j=i+1}^g c'_j - c_j) - M},$$

with every tensor exponent nonnegative. Then the morphism  $\mathcal{L}_{\text{md}(w')} \rightarrow \mathcal{L}_{\text{md}(w)}$  vanishes precisely where the tensor exponents are strictly positive, which is the definition of having  $\epsilon_{w',w}^i = 0$ .

For the second assertion, the  $w$  yielding multidegree concentrated on  $Z_i$  is given by  $(c_2, \dots, c_g)$  with  $c_{i'} = 0$  for  $i' \leq i$  and  $c_{i'} = d'$  for  $i' > i$ . Thus, if  $0 \leq c'_{i'} \leq d'$  for all  $i'$ , we have that  $c'_{i'} - c_{i'} \leq 0$  for  $i' > i$  and  $c'_{i'} - c_{i'} \geq 0$  for  $i' \leq i$ , so  $\sum_{j=i'+1}^g (c'_j - c_j)$  achieves its minimum at  $i' = i$ , and hence  $\epsilon_{w',w}^i = 1$  in this case.  $\square$

We summarize what are for us the key properties of chain-adaptable limit linear series as follows:

**Proposition 4.7.** *Let  $(\mathcal{L}^i, V^i)$  be a chain-adaptable limit linear series, with  $s_j^i$  as in the definition. Choose also  $s_0^1, \dots, s_r^1$  in  $V^1$  satisfying  $\text{ord}_{Q_1} s_r^1 < \text{ord}_{Q_1} s_{r-1}^1 < \dots < \text{ord}_{Q_1} s_0^1$  and  $s_0^n, \dots, s_r^n$  in  $V^n$  satisfying  $\text{ord}_{P_n} s_0^n < \text{ord}_{P_n} s_1^n < \dots < \text{ord}_{P_n} s_r^n$ . For  $j = 0, \dots, r$ , set  $w_j = (\text{ord}_{P_2} s_j^2, \dots, \text{ord}_{P_g} s_j^g)$ .*

*Then for  $j = 0, \dots, r$ , there exists  $s_j \in \Gamma(X_0, \mathcal{L}_{\text{md}(w_j)})$  such that for each  $i$ , we have that  $s_j|_{Z_i}$  agrees with  $s_j^i$  up to scalar. Moreover, for each  $w_j$ , the subspace of  $\Gamma(X_0, \mathcal{L}_{\text{md}(w_j)})$  consisting of sections restricting to  $V^i$  on  $Z_i$  for all  $i$  has dimension precisely  $r+1$ .*

Indeed, the above is essentially Propositions 5.2.3 and 5.2.6 of [Oss14], although the situation is simpler in our present rank-1 setting.

We now move from considering individual reducible curves to degenerating families of curves. At the same time, we temporarily return to the general setting of curves of compact type, because the relevant arguments are not special to chains of curves. Thus, we adopt the following situation.

**Situation 4.8.** Let  $\pi : X \rightarrow B$  be a flat, proper morphism, with 1-dimensional fibers, and  $B$  the spectrum of a discrete valuation ring. Suppose further that  $X$  is regular, the generic fiber  $X_\eta$  is smooth, and the special fiber  $X_0$  is nodal, of compact type, with dual graph  $V(\Gamma)$ , and components  $Z_v$  for  $v \in V(\Gamma)$ . For each  $v \in V(\Gamma)$ , let  $\hat{\sigma}_v \in \Gamma(X, \mathcal{O}_X(Z_v))$  be a section vanishing precisely on  $Z_v$ . Also, choose an isomorphism

$$\hat{\theta} : \bigotimes_{v \in V(\Gamma)} \mathcal{O}_X(Z_v) \xrightarrow{\sim} \mathcal{O}_X.$$

We then see that  $\mathcal{O}_X(Z_v)$  and  $\hat{\sigma}_v$  induce systems of line bundles and sections as we had previously constructed on  $\sigma_v$ . Indeed, the following is then almost immediate.

**Proposition 4.9.** *For  $v \in V(\Gamma)$ , we have  $\mathcal{O}_X(Z_v)|_{X_0} \cong \mathcal{O}^v$ , and  $\hat{\sigma}_v|_{X_0}$  is a valid choice of  $\sigma_v$ . Similarly,  $\hat{\theta}|_{X_0}$  is a valid choice of  $\theta$ .*

Recall that without the family of curves,  $\sigma_v$  was not unique up to scalar, but we see from the above that our family induces a choice of  $\sigma_v$ . This is one way in which direction of approach can be incorporated into our analysis.

Now, suppose we have a flat base change  $B' \rightarrow B$  with  $B'$  still the spectrum of a discrete valuation ring. This thus induces  $\pi' : X' \rightarrow B'$  with a special fiber  $X'_0$  which is simply a base extension of  $X_0$ , and generic fiber  $X'_\eta$  which is likewise a base change of  $X_\eta$ . Suppose we have a linear series  $(\mathcal{L}_\eta, V_\eta)$  of rank  $r$  and degree  $d$  on  $X'_\eta$ . By the compact type hypothesis, we know that for every multidegree  $\omega$  of total degree  $d$ , there is a unique extension  $\overline{\mathcal{L}}$  of  $\mathcal{L}_\eta$  over all  $X'$  such that the restriction to  $X'_0$  has multidegree  $\omega$ ; denote this by  $\overline{\mathcal{L}}_\omega$ . We can construct a system of choices of the  $\overline{\mathcal{L}}_\omega$  together with morphisms between them just as we did above, with (the pullbacks to  $X'$  of)  $\mathcal{O}_X(Z_v)$  and  $\hat{\sigma}_v$  in place of  $\mathcal{O}^v$  and  $\sigma_v$ , and  $\hat{\theta}$  in place of  $\theta$ . Then given an extension  $\overline{\mathcal{L}}_\omega$ , we also obtain an extension  $\overline{V}_\omega$  simply by taking

$$\overline{V}_\omega = V_\eta \cap \Gamma(X', \overline{\mathcal{L}}_\omega) \subseteq \Gamma(X'_\eta, \mathcal{L}_\eta).$$

From the definition of this extension, we see immediately that both it and the corresponding quotient are torsion-free, hence free. A key observation for us (initially developed in [Oss06]) is that for any multidegrees  $\omega, \omega'$ , we have that  $\overline{V}_{\omega'}$  maps into  $\overline{V}_\omega$  under the above-constructed morphism  $\overline{\mathcal{L}}_{\omega'} \rightarrow \overline{\mathcal{L}}_\omega$ .

We finally turn to the specific subject of interest: multiplication maps. Accordingly, we introduce the following additional data.

**Situation 4.10.** In Situation 4.8, suppose we are given also a base change  $\pi' : X' \rightarrow B'$  as above, and let  $(\mathcal{L}_1, V_1), \dots, (\mathcal{L}_m, V_m)$  be linear series (possibly of different ranks and degrees) on  $X'_\eta$ .

Our objective is to study the multiplication map

$$\mu : V_1 \otimes \cdots \otimes V_m \rightarrow \Gamma(X'_\eta, \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_m)$$

by considering how it limits to  $X'_0$ . For each  $\mathcal{L}_i$ , we fix systems of extensions  $\overline{\mathcal{L}}_{i, \omega_i}$  as above for each multidegree  $\omega_i$  of total degree equal to  $\deg \mathcal{L}_i$ , and if we set  $\mathcal{L} := \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_m$ , we also fix a system of extensions  $\overline{\mathcal{L}}_\omega$  of  $\mathcal{L}$  for each  $\omega$  of total degree equal to  $\sum_i \deg \mathcal{L}_i$ . Now, as discussed above we can extend each  $V_i$  in multidegree  $\omega_i$  by setting

$$\overline{V}_{i, \omega_i} := V_i \cap \Gamma(X', \overline{\mathcal{L}}_{i, \omega_i}),$$

and similarly, if we write  $W_\eta$  for the image of  $\mu$ , then  $(\mathcal{L}, W_\eta)$  is itself a linear series, so we can extend it to

$$\overline{W}_\omega := W_\eta \cap \Gamma(X', \overline{\mathcal{L}}_\omega).$$

If we choose any  $\omega_i$ s, and set  $\omega = \sum_i \omega_i$ , we can also extend our multiplication map to obtain

$$\overline{\mu} : \overline{V}_{1, \omega_1} \otimes \cdots \otimes \overline{V}_{m, \omega_m} \rightarrow \Gamma(X', \overline{\mathcal{L}}_\omega).$$

We see immediately from the construction that the image of  $\overline{\mu}$  is contained in  $\overline{W}_\omega$ . Because reduction to the special fiber is surjective, we likewise have that the image of the restriction of  $\overline{\mu}$  to  $X'_0$  is contained in the restriction of  $\overline{W}_\omega$ . Finally, given multidegrees  $\omega, \omega'$ , as we observed above we have that  $\overline{W}_{\omega'}$  maps into  $\overline{W}_\omega$  under our constructed maps.

To summarize, if we restrict to the special fiber, we have a system of spaces  $\overline{W}_\omega|_{X'_0}$ , each of dimension equal to  $\dim W_\eta$ , containing the images of the appropriate multiplication maps  $\overline{\mu}|_{X'_0}$  and linked together by natural maps. So if we have a tuple of sections  $s_1, \dots, s_m$  in  $\overline{V}_{1, \omega_1}|_{X'_0}, \dots, \overline{V}_{m, \omega_m}|_{X'_0}$ , and set  $\omega' = \sum_i \omega_i$ , then  $s_1 \otimes \cdots \otimes s_m$  is in  $\overline{W}_{\omega'}|_{X'_0}$ . If we fix a multidegree  $\omega$ , the image of  $s_1 \otimes \cdots \otimes s_m$  under the constructed map from multidegree  $\omega'$  to multidegree  $\omega$  lies in  $\overline{W}_\omega|_{X'_0}$ . Our strategy is then to construct many such sections in different multidegrees, and consider all of their images inside a single multidegree  $\overline{W}_\omega|_{X'_0}$ . If we can show that the images span a space of dimension  $N$ , then this implies that  $\overline{W}_\omega|_{X'_0}$  has dimension at least  $N$ , and hence that  $W_\eta$  had dimension at least  $N$  as well.

From now on, we restrict to the case of interest in the Maximal Rank Conjecture, where  $\mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_m$ . From the above discussion, we will be able to conclude the following criterion.

**Corollary 4.11.** *Given  $(g, r, d, m)$  as in Definition 1.3, suppose  $Z_1, \dots, Z_n$  are smooth projective curves, with  $g$  of the  $Z_i$  having genus 1, and the rest having genus 0. Suppose also that  $P_i, Q_i$  are marked points of  $Z_i$  for each  $i$ , such that for all  $i$  with  $Z_i$  of genus 1, we have that  $P_i - Q_i$  is not  $\ell$ -torsion for any  $\ell \leq d$ . Let  $X_0$  be the genus- $g$  nodal curve obtained from the  $Z_i$  by gluing  $Q_i$  to  $P_{i+1}$  for  $i = 1, \dots, n-1$ . Suppose we have a chain-adaptable limit linear series  $(\mathcal{L}^i, V^i)$  on  $X_0$  of rank  $r$  and degree  $d$  such that, if  $s_j \in \Gamma(X_0, \mathcal{L}_{\text{md}(w_j)})$  are the global sections arising from the chain-adaptability condition, there exists a  $w \in \mathbb{Z}^{n-1}$  such that for all choices of the sections  $\sigma_i$  as above, the images of the  $s_{j_1} \otimes \cdots \otimes s_{j_m}$  in  $\Gamma(X_0, (\mathcal{L}^{\otimes m})_{\text{md}(w)})$  have at least  $N$ -dimensional span.*

*Then for any smoothing  $\pi : X \rightarrow B$  of  $X_0$  as in Situation 4.8, the generic fiber of the smoothing family is a smooth genus- $g$  curve  $X$  which carries a linear series  $(\mathcal{L}, V)$  of rank  $r$  and degree  $d$  on  $X$  such that the  $m$ -multiplication map (1.2) for  $V$  has rank at least  $N$ .*

*If further  $n = g$  and we have  $(w_{j_1} + \cdots + w_{j_m}, w)$  steady for all  $(j_1, \dots, j_m)$ , then  $X_0$  is not in the closure of the locus in  $\mathcal{M}_g$  corresponding to curves which do not carry an  $(\mathcal{L}, V)$  having  $m$ -multiplication map of rank at least  $N$ .*

*Proof.* First, the condition that  $P_i - Q_i$  is not  $\ell$ -torsion for  $\ell \leq d$  implies that the space of limit linear series on  $X_0$  has expected dimension  $\rho$ . This implies that if  $\pi : X \rightarrow B$  is any regular smoothing family of  $X_0$ , every limit linear series on  $X_0$  is a limit of linear series on the smooth fibers of  $\pi$ : that is, there exists a flat base change  $B' \rightarrow B$  and an  $(\mathcal{L}, V)$  on the generic fiber of  $X' := X \times_B B'$  such

that  $(\mathcal{L}, V)$  extends as described above to the chosen limit linear series. Indeed, since refinedness is part of the definition of chain adaptability, this follows from the original Eisenbud-Harris smoothing theorem (Corollary 3.5 of [EH86]).

Now, let  $W$  denote the image of  $V^{\otimes m}$  under multiplication, so that we want to prove that  $W$  has dimension at least  $N$ . Then we first observe that each section  $s_j$  must be in the multidegree- $\text{md}_d(w_j)$  limit of  $(\mathcal{L}, V)$ : indeed, the limit of  $V$  has dimension  $r + 1$  and maps into the  $V^i$  under each restriction map, so according to the second part of Proposition 4.7, we have that the limit of  $V$  is the entire subspace of global sections of  $\mathcal{L}_{\text{md}(w_j)}$  which restricts into  $V^i$  on  $Z_i$  for all  $i$ , and in particular it contains  $s_j$ . Thus, we likewise have that each  $s_{j_1} \otimes \cdots \otimes s_{j_m}$  is in the multidegree- $\text{md}_{md}(w_{j_1} + \cdots + w_{j_m})$  limit of  $(\mathcal{L}^{\otimes m}, W)$ . Then it also follows from the above discussion that the image of  $s_{j_1} \otimes \cdots \otimes s_{j_m}$  in multidegree  $\text{md}_{md}(w)$  lies in the multidegree- $\text{md}_{md}(w)$  limit of  $(\mathcal{L}^{\otimes m}, W)$ . If, as the  $(j_1, \dots, j_m)$  vary, these images span a space of dimension  $N$ , then it follows that  $W$  has dimension at least  $N$ , as desired. This proves the first assertion of the corollary.

In order to prove the stronger statement under the additional steadiness hypothesis, we carry out a similar analysis when the smoothing family  $\pi : X \rightarrow B$  is not assumed regular. Because we have also assumed  $n = g$ , such families can be used to study arbitrary curves in  $\overline{\mathcal{M}}_g$  specializing to  $X_0$ . In this situation we can blow up  $X$  to obtain a regular family  $\tilde{\pi} : \tilde{X} \rightarrow \tilde{B}$ , where the special fiber  $\tilde{X}_0$  is obtained from  $X_0$  by inserting (possibly empty) chains of projective lines at the nodes of  $X_0$ . It suffices to show that in this case the hypotheses of the corollary apply equally to  $\tilde{X}_0$ , since we can then apply the first part of the corollary to conclude the desired statement for the family  $\tilde{\pi}$ , whose smooth fibers agree with those of  $\pi$ . It is clear what limit linear series we should choose: if we insert a projective line with marked points  $P, Q$  at a node which has vanishing on one side given by  $b_r, \dots, b_0$  and on the other by  $a_0, \dots, a_r$ , so that  $a_j + b_{r-j} = d$  for all  $j$ , then we have sections of  $\mathcal{O}(d)$ , unique up to scaling, with vanishing order  $a_j$  at  $P$  and  $b_{r-j}$  at  $Q$ . If we take the span of these  $r + 1$  sections, we then obtain a  $\mathfrak{g}_d^r$  on the projective line, and if we repeat this procedure for every inserted projective line, we will obtain a new chain-adaptable limit linear series on  $\tilde{X}_0$ . If we  $\tilde{X}_0$  has  $n'$  components, this limit linear series has corresponding global sections  $\tilde{s}_0, \dots, \tilde{s}_r$  in multidegrees determined by  $\tilde{w}_0, \dots, \tilde{w}_r \in \mathbb{Z}^{n'-1}$ , where  $\tilde{w}_j$  is obtained from the  $w_j$  by adding repetitions at every inserted projective line (in the usual notion of multidegree, we will assign multidegree 0 to every inserted component). We then apply the same procedure to  $w$  to obtain a  $\tilde{w}$ .

By construction, the sections  $\tilde{s}_j$  agree with  $s_j$  (at least, up to scalar) after restriction to any given component of  $X_0$ , so the same applies to their tensor products  $\tilde{s}_{\vec{j}}$  for any  $\vec{j} = (j_1, \dots, j_m)$ . Now, in general the insertion of the new components can change which components are zeroed out in mapping from multidegree  $\text{md}_{md}(\tilde{w}_{\vec{j}})$  to multidegree  $\text{md}_{md}(\tilde{w})$ , even on the components of  $\tilde{X}_0$  coming from  $X_0$ . Indeed, the sums  $\sum_{i'=i+1}^g (c'_{i'} - c_{i'})$  appearing in the definition of  $\epsilon_{(\tilde{w}_{\vec{j}}, \tilde{w})}^i$  will have some extra repetitions inserted corresponding to the new components. If one has  $i < i'$  such that  $c'_i < c_i$  and  $c'_{i'} > c_{i'}$ , inserting repetitions can change where the minimum is achieved. However, this is precisely ruled out by the steadiness hypothesis, so we see that with this hypothesis, we will have the map from multidegree  $\text{md}_{md}(\tilde{w}_{\vec{j}})$  to multidegree  $\text{md}_{md}(\tilde{w})$  nonzero precisely on the components  $Z_i$  on which the

original map was nonzero, together with any inserted components connecting two components on which the map is nonzero. We conclude that on each component of  $\tilde{X}_0$  coming from  $X_0$ , the image of  $\tilde{s}_{\vec{j}}$  in multidegree  $\text{md}_{md}(\tilde{w})$  agrees up to scalar with the image of  $s_{\vec{j}}$  in multidegree  $\text{md}_{md}(\tilde{w})$ . Now, observe that since  $\tilde{w}$  induces multidegree 0 on each inserted projective line, we have a canonical ‘contraction’ isomorphism

$$\Gamma(\tilde{X}_0, \mathcal{L}_{\text{md}(\tilde{w})}) \xrightarrow{\sim} \Gamma(X_0, \mathcal{L}_{\text{md}(w)}),$$

and we see that under this isomorphism, the images of the  $\tilde{s}_{\vec{j}}$  in multidegree  $\text{md}_{md}(\tilde{w})$  agree up to scalar with the images of the  $s_{\vec{j}}$  in multidegree  $\text{md}_{md}(w)$ . Indeed, this follows from the steadiness hypothesis, which ensures that not only do the sections in question agree up to scalar after restriction to each component of  $X_0$ , but their support is a contiguous collection of components  $Z_i \cup \dots \cup Z_{i'}$  for some  $i' \geq i$ , and the sections do not vanish at any of the nodes  $Q_i, \dots, Q_{i'-1}$ . We conclude that if the images of the  $\tilde{s}_{\vec{j}}$  in multidegree  $\text{md}_{md}(\tilde{w})$  span a space of dimension at least  $N$ , the same is true of the images of the  $s_{\vec{j}}$  in multidegree  $\text{md}_{md}(w)$ . Thus, our hypotheses on the limit linear series on  $X_0$  imply that the same hypotheses are satisfied on  $\tilde{X}_0$ , as desired. The corollary follows.  $\square$

We can finally complete the proof of our elementary criterion.

*Proof of Theorem 3.9.* First, note that the definition of  $(g, r, d)$ -sequence implies that for each  $i$ , the values of  $a_j^i$  are strictly increasing and between 0 and  $d$ , so that the values of  $b_j^i$  are strictly decreasing and also between 0 and  $d$ .

Let  $Z_1, \dots, Z_g$  be nonsupersingular genus-1 curves, and  $P_i, Q_i$  marked points on each  $Z_i$ . Suppose further that the  $P_i, Q_i$  do not differ by  $\ell$ -torsion for any  $\ell \leq d$ . Let  $X_0$  be the nodal curve obtained by gluing  $Q_i$  to  $P_{i+1}$  for  $i = 1, \dots, g-1$ . Following the notation of Definition 3.4, we will use the table  $T'(\vec{\delta})$  to define a limit linear series on  $X_0$ , together with sections on each aspect, as follows: for  $i = 1, \dots, g$ , let  $\mathcal{L}^i = \mathcal{O}_{Z_i}(a_{\delta_i}^i P_i + b_{\delta_i}^i Q_i)$ , so that  $\mathcal{L}^i$  has degree  $d$ . Then for  $j = 0, \dots, r$ , let  $s_j^i$  be a section of  $\mathcal{L}^i$  vanishing to order at least  $a_j^i$  at  $P_i$  and at least  $b_j^i$  at  $Q_i$ ; these exist and are unique up to scalar because for  $j \neq \delta_i$ , we have  $a_j^i + b_j^i = d - 1$ . Moreover, by our non-torsion hypothesis, the orders of vanishing at  $P_i$  and  $Q_i$  are precisely  $a_j^i$  and  $b_j^i$  respectively. These sections are linearly independent since they have distinct vanishing orders at  $P_i$ , so if we set  $V^i = \text{span}(s_0^i, \dots, s_r^i)$ , we have that  $(\mathcal{L}^i, V^i)$  is a  $\mathbf{g}_d^r$  on  $Z_i$ . Moreover, the collection of  $(\mathcal{L}^i, V^i)$  by construction form a refined limit linear series on  $X_0$ , which is moreover chain-adaptable. Consequently, for  $j = 0, \dots, r$  we can set  $w_j' = (a_j^2, \dots, a_j^g)$  and we have a section  $s_j \in \Gamma(X_0, \mathcal{L}_{\text{md}(w_j')})$  such that for each  $i$ , we have  $s_j|_{Z_i}$  agreeing with  $s_j^i$  up to scalar.

Now, for  $\vec{j} = j_1 \leq \dots \leq j_m$ , we can form

$$s_{\vec{j}} = s_{j_1} \otimes \dots \otimes s_{j_m} \in \Gamma(X_0, (\mathcal{L}^{\otimes m})_{w_{\vec{j}}'}),$$

where  $w_{\vec{j}}' = w_{j_1}' + \dots + w_{j_m}'$ . Finally, if we fix any choices of  $\sigma_1, \dots, \sigma_g$  as in Situation 4.2, then using the natural map from multidegrees  $\text{md}_{md}(w_{\vec{j}}')$  to  $\text{md}_{md}(w)$ , we obtain the image  $s_{\vec{j}, w}$  of  $s_{\vec{j}}$  in multidegree  $\text{md}_{md}(w)$ . By Proposition 4.6, this will be zero on  $Z_i$  precisely when  $\epsilon_{w_{\vec{j}, w}}^i = 0$ , and will otherwise still have  $s_{\vec{j}, w}|_{Z_i}$  agreeing

with  $s_{j_1}^i \otimes \cdots \otimes s_{j_m}^i$  up to scalar; in particular, when  $\epsilon_{w',w}^i = 1$ , we have that  $s_{\vec{j},w}|_{Z_i}$  vanishes to order  $a_{\vec{j}}^i$  at  $P_i$  and to order  $b_{\vec{j}}^i$  at  $Q_i$  (considered as sections of  $\mathcal{L}^i$ ).<sup>1</sup>

Suppose that  $T_w(\vec{\delta})$  is  $N$ -expungeable: we claim that the  $s_{\vec{j},w}$  have at least  $N$ -dimensional span. This will prove the theorem, by virtue of Corollary 4.11. More specifically, we claim that the selected  $N$  rows of  $T_w(\vec{\delta})$  correspond to linearly independent  $s_{\vec{j},w}$ . Suppose we have a linear dependence  $\sum_{\vec{j}} \gamma_{\vec{j}} s_{\vec{j},w} = 0$ ; we apply the rules in the definition of expungeable to successively show that some of the  $\gamma_{\vec{j}}$  must be 0, so that if the expungeability condition holds, we have shown all  $\gamma_{\vec{j}} = 0$ , as desired. We thus successively examine the rules for expungeability, starting with (ii). In this case, the hypothesis of rule (ii) says that among the  $s_{\vec{j}',w}$  which are nonzero on  $Z_i$  and for which we have not already shown  $\gamma_{\vec{j}'} = 0$ , we have  $\text{ord}_{P_i} s_{\vec{j},w}|_{Z_i} < \text{ord}_{P_i} s_{\vec{j}',w}|_{Z_i}$  whenever  $\vec{j}' \neq \vec{j}$ . Thus, we cannot have  $\gamma_{\vec{j}} \neq 0$  in a linear dependence. Rule (iii) is the same, with  $Q_i$  in place of  $P_i$ . The one-row case of Rule (iv) is clear, since if there is a unique  $\vec{j}$  for which we haven't already shown  $\gamma_{\vec{j}} = 0$  such that  $s_{\vec{j},w}|_{Z_i} \neq 0$ , it immediately follows that  $\gamma_{\vec{j}} = 0$ . On the other hand, the two-row case is a special case of Rule (v), since given  $\vec{j} \neq \vec{j}'$ , we must have some  $j \neq \delta_i$  which occurs a different number of times in  $\vec{j}$  and  $\vec{j}'$ .

For Rule (v), if  $j \neq \delta_i$ , we have  $\text{div } s_j^i = a_j^i P_i + b_j^i Q_i + R_j^i$  for a uniquely determined  $R_j^i$ . Under our non-torsion hypothesis, all the  $R_j^i$  are distinct, and the hypothesis of Rule (v) then implies that  $\text{ord}_{R_j^i} s_{\vec{j},w}|_{Z_i} = n$ , while  $\text{ord}_{R_j^i} s_{\vec{j}',w}|_{Z_i} > n$  for all remaining  $\vec{j}' \neq \vec{j}$  for which we have not already shown  $\gamma_{\vec{j}'} = 0$ . We thus conclude  $\gamma_{\vec{j}} = 0$  as in Rules (ii) and (iii).

We now prove Rule (vi). The conditions on the  $a_{j_e}^i$  and  $b_{j_e}^{i+1}$  imply that if a linear dependence has nonzero coefficients  $\gamma_{\vec{j}_1}$  and  $\gamma_{\vec{j}_2}$  for  $s_{\vec{j}_1}$  and  $s_{\vec{j}_2}$ , then the leading terms of  $\gamma_{\vec{j}_1} s_{\vec{j}_1}$  and  $\gamma_{\vec{j}_2} s_{\vec{j}_2}$  must cancel at both  $P_i$  and  $Q_{i+1}$ . Note also that our hypotheses on the  $\vec{j}^e$  imply that  $b_{j_1}^i = b_{j_2}^i$  (they must either both be equal to  $md - a_{j_e}^i - 2$  or to  $md - a_{j_e}^i - m$ ), and thus that  $a_{j_1}^{i+1} = a_{j_2}^{i+1}$  as well. It thus makes sense to normalize our scaling of  $s_{\vec{j}_1}$  and  $s_{\vec{j}_2}$  so that their values agree at  $Q_i$  (equivalently, at  $P_{i+1}$ ). First suppose that all but exactly two of the  $j_n^e$  are equal to  $\delta_i$  for both  $e = 1$  and  $e = 2$ , and  $a_{j_1}^i \neq a_{(\delta_i, \dots, \delta_i)}^i - 1$ . In this case, with the stated normalization, and a given choice of  $P_{i+1}, Q_{i+1}$ , the desired cancellation at  $Q_{i+1}$  will determine a unique ratio for  $\gamma_{\vec{j}_1}$  and  $\gamma_{\vec{j}_2}$ . It suffices then to show that if we vary  $P_i, Q_i$ , the ratio determined by cancellation at  $P_i$  varies nontrivially. But note that the  $m - 2$  copies of  $s_{\delta_i}^i$  in  $s_{\vec{j}_1}^i$  and  $s_{\vec{j}_2}^i$  do not affect this variation, so this follows from Corollary 2.4. The other case follows similarly, except that we fix  $P_i, Q_i$  and consider the effects of letting  $P_{i+1}, Q_{i+1}$  vary.

<sup>1</sup>There is a technical point here to address: typically, we have  $0 \leq c_{i'} \leq md$  for all  $i'$ , so that according to the second part of Proposition 4.6, the natural map  $\mathcal{L}_{\text{md}(w)} \rightarrow \mathcal{L}^i$  is nonzero. However, it is sometimes convenient to allow some of the  $c_{i'}$  to violate the preceding inequality, in which case the map  $\mathcal{L}_{\text{md}(w)} \rightarrow \mathcal{L}^i$  may vanish identically. Luckily, this issue is easily addressed: choose  $a \geq 0$  so that  $-a \leq c_{i'} \leq md + a$  for all  $i'$ , and let  $w^i = (-a, \dots, -a, md + a, \dots, md + a)$ , with  $i - 1$  entries equal to  $-a$ . Then by the same argument as in the second part of Proposition 4.6, the natural maps  $\mathcal{L}_{\text{md}(w)} \rightarrow \mathcal{L}_{\text{md}(w^i)}|_{Z_i}$  and  $\mathcal{L}^i \rightarrow \mathcal{L}_{\text{md}(w^i)}|_{Z_i}$  are both nonzero, and we may therefore consider the images of everything in  $\mathcal{L}_{\text{md}(w^i)}|_{Z_i}$  instead of in  $\mathcal{L}^i$ , which will not affect any of our arguments.

Finally, we consider Rule (vii): if we let  $Z_I = \cup_{i'=i}^{i+n-1} Z_{i'}$ , we wish to show that with the given hypotheses, we cannot have a nontrivial linear relation

$$\gamma_{\vec{j}_1} s_{\vec{j}_1}|_{Z_I} + \cdots + \gamma_{\vec{j}_n} s_{\vec{j}_n}|_{Z_I} + \gamma_{(\delta_i, \delta_i)} s_{(\delta_i, \delta_i)}|_{Z_I} = 0.$$

Observe that the conditions  $\delta_{i'} = \delta_i$  and  $j_1^e < \delta_i < j_2^e$  imply that  $a_{j_e}^{i'} = a_{j_e}^i + 2(i' - i)$  for  $i' = i, \dots, i + n - 1$ , and in particular  $a_{j_e}^{i'}$  is also independent of  $e$  for  $i' > i$ , and is equal to  $c_{i'}$ .

Now, our linear dependence, if nonzero, must cancel all leading terms at all  $P_{i'}$  and  $Q_{i'}$ ; cancellation at a given  $Q_{i'}$  is equivalent to cancellation at  $P_{i'+1}$ , so this works out to at most  $n + 1$  conditions. The rough idea of our argument is we will see that when the chosen marked points are general, we always obtain either  $n + 1$  or  $n$  conditions in this way, and the latter occurs in a situation where  $s_{(\delta_i, \delta_i)}$  never contributes to the leading terms. More specifically, we proceed from  $i' = i$  to  $i' = i + n - 1$ , showing that if we fix the previous choices of  $P_{i'}$ ,  $Q_{i'}$ , a general choice of the current  $Q_{i'}$  will impose an additional linear condition on the choice of the  $\gamma_{\vec{j}}$ , with at most one exception. This is slightly delicate, because the  $s_{\vec{j}}$  are only defined up to independent scaling, which changes the spaces of linear dependencies, but we can address this as follows: when we want to consider the effect of different positions of  $Q_{i'}$ , we will normalize so that each  $s_{\vec{j}_e}$  has the same value at  $P_{i'}$ . This makes sense because we have observed that  $a_{j_e}^{i'}$  is independent of  $e$ .

We need some preliminary observations on when the  $(\delta_i, \delta_i)$  row can appear in a given column. Note that for every  $i' = i, \dots, i + n - 1$ , the entries of the  $(\delta_i, \delta_i)$  row in the  $i'$  column add up to  $2d$ , whereas the entries of the  $\vec{j}^e$  row add up to  $2d - 2$ . Let  $i_0$  be the smallest number between  $i$  and  $i + n - 1$  such that  $a_{(\delta_i, \delta_i)}^{i_0} \geq c_{i_0}$  and  $b_{(\delta_i, \delta_i)}^{i_0} \geq 2d - c_{i_0+1}$ , so that  $s_{(\delta_i, \delta_i)}$  may appear in column  $i_0$ . First observe that if there is any  $i'$  with  $a_{j_e}^{i'} = a_{(\delta_i, \delta_i)}^{i'} - 1$ , then we have also  $b_{j_e}^{i'} = b_{(\delta_i, \delta_i)}^{i'} - 1$ , and furthermore  $i_0 = i'$  is the only column between  $i$  and  $i + n - 1$  in which  $s_{(\delta_i, \delta_i)}$  may occur. Similarly, if for some  $i'$  we have  $b_{j_e}^{i'} = b_{(\delta_i, \delta_i)}^{i'}$ , then  $a_{j_e}^{i'} = a_{(\delta_i, \delta_i)}^{i'} - 2$ , so we must have  $i_0 = i'$ . In this case, if  $i' < i + n - 1$  we can also have  $s_{(\delta_i, \delta_i)}$  occurring in the next column, but not in any others, since we will have  $b_{j_e}^{i'+1} = b_{(\delta_i, \delta_i)}^{i'+1} - 2$ .

We now begin our analysis with the case  $i' = i$ : let  $W_i \subseteq k^n$  be the subspace of choices of  $\gamma_{\vec{j}_e}$  such that there exists a  $\gamma_{(\delta_i, \delta_i)}$  giving a valid linear dependence on  $Z_i$ . If  $i_0 > i$ , then cancellation of lowest-order terms at  $P_i$  is a codimension-1 subspace  $H \subseteq k^n$  containing  $W_i$  (specifically, given our normalization, it is the hyperplane  $\sum_i x_i = 0$ ). Moreover, when  $i_0 > i$  we have observed above that  $a_{j_e}^i \neq a_{(\delta_i, \delta_i)}^i - 1$ , so under our normalization hypotheses, the sections  $(s_{\vec{j}_1}^i|_{Z_i}, \dots, s_{\vec{j}_n}^i|_{Z_i})$  satisfy the hypotheses of Corollary 2.7, and the map

$$Q_i \mapsto (s_{\vec{j}_1}^i|_{Z_i}(Q_i), \dots, s_{\vec{j}_n}^i|_{Z_i}(Q_i))$$

is nondegenerate. In particular, it is nonconstant, so a general choice of  $Q_i$  will not have image not equal to (the projectivization of) the orthogonal complement of  $H$ , meaning that imposing cancellation of lowest order terms at  $Q_i$  will impose a different codimension-1 condition. In this case, we thus have that  $W_i$  is at most  $(n - 2)$ -dimensional.



On the other hand, if  $i_0 = i$ , we claim that  $W_i$  has dimension at most  $n - 1$ . Indeed, if  $a_{j^e}^i > a_{(\delta_i, \delta_i)}^i$ , then  $a_{(\delta_i, \delta_i)}^i$  is a unique minimum, so by Rule (ii) we can drop the  $(\delta_i, \delta_i)$  row, and we are in the same situation as above, with  $\dim W_i = n - 2$ . On the other hand, if  $a_{j^e}^i < a_{(\delta_i, \delta_i)}^i$ , then  $W_i$  is still contained in the hyperplane  $H$  described above. Finally, if  $a_{j^e}^i = a_{(\delta_i, \delta_i)}^i$ , then  $b_{j^e}^i = b_{(\delta_i, \delta_i)}^i - 2$ , and in this case  $W_i$  is contained in the hyperplane obtained by looking at cancellation of the leading coefficients at  $Q_i$ .

Now, for  $i' > i$ , let  $W_{i'-1} \subseteq k^n$  be the subspace of choices of  $\gamma_{\vec{j}}$  such that there exists a choice of  $\gamma_{(\delta_i, \delta_i)}$  giving a valid linear dependence on  $Z_i \cup \dots \cup Z_{i'-1}$ . If  $W_{i'-1} = 0$ , we are done. Otherwise, our inductive hypothesis is that  $W_{i'-1}$  has codimension at least  $i' - i + 1$  if  $i_0 \geq i'$ , and  $W_{i'-1}$  has codimension at least  $i' - i$  if  $i_0 < i'$ . We then want to show that imposing linear dependence also on  $Z_{i'}$  reduces the dimension of  $W_{i'-1}$  by 1 unless  $i' = i_0$ . First, if we have either  $a_{j^e}^{i'} = a_{(\delta_i, \delta_i)}^{i'} - 1$  or  $b_{j^e}^{i'} = b_{(\delta_i, \delta_i)}^{i'}$ , then necessarily  $i' = i_0$ ; in this case, there is nothing to show. So we can assume that  $a_{j^e}^{i'} \neq a_{(\delta_i, \delta_i)}^{i'} - 1$  and  $b_{j^e}^{i'} \neq b_{(\delta_i, \delta_i)}^{i'}$ . The latter means that in order to have linear dependence on  $Z_{i'}$ , we must have cancellation among the leading coefficients at  $Q_{i'}$  of the  $s_{j^e}$ , and the former implies that, just as in the case  $i' = i$ , we have that the map

$$Q_{i'} \mapsto (s_{j_1}^{i'}|_{Z_{i'}(Q_{i'})}, \dots, s_{j_n}^{i'}|_{Z_{i'}(Q_{i'})})$$

is nondegenerate. In particular, a general choice of  $Q_{i'}$  will have image not lying in the orthogonal complement of  $W_{i'-1}$ , meaning that requiring that the  $\gamma_{\vec{\delta}}$  impose a linear dependence also on  $Z_{i'}$  reduces the dimension of  $W_{i'-1}$  by 1, as desired.

Because  $Z_I$  has  $n$  components, we thus conclude that when we have imposed cancellation of leading terms at all  $P_{i'}$  and  $Q_{i'}$ , we have reduced the space of possible linear dependences to (0), proving Rule (vii).

We thus conclude that the  $s_{\vec{j}, w}$  have at least  $N$ -dimensional span, and applying Corollary 4.11 we obtain Theorem 3.9.  $\square$

*Remark 4.12.* In the absence of the steadiness condition, the images of the  $s_{\vec{j}}$  in a given multidegree  $\text{md}_{md}(w)$  may depend on the smoothing of the given curve in two distinct ways. First, they may depend on the choice of maps  $\sigma_i$ . This arises when the non-erased portions of some rows of  $T_w(\vec{\delta})$  are non-contiguous; in this case, modifying the  $\sigma_i$  may scale different parts of the image of  $s_{\vec{j}}$  independently, potentially affecting the dimension of the span. Second, by considering nonregular smoothings and inserted  $\mathbb{P}^1$ s as in the proof of Corollary 4.11, it is possible to change which parts of  $T_w(\vec{\delta})$  are erased in any row in which steadiness fails.

This could become important if it turns out that there are some  $(g, r, d, m)$  such that every chain of genus-1 lies in the closure of the locus in  $\mathcal{M}_g$  for which the maximal rank condition fails. Indeed, both of the phenomena described above can be used to generalize the notion of expungeability, yielding criteria for proving the Maximal Rank Conjecture which, by requiring restrictions on direction of smoothing, could apply even in such cases. However, we have not yet found examples to which these generalized criteria apply which we cannot handle using Theorem 3.9, so in order to avoid unnecessary complication we have developed our approach without any techniques involving direction of approach.

*Remark 4.13.* We describe two ways in which the usual limit linear series approach to multiplication maps fails already in the example  $m = 2, r = 4, g = 10, d = 12$  treated in Example 3.13. First, we could try to prove injectivity by looking at orders of vanishing of the limit of a hypothetical element of the kernel of (1.2). However, as pointed out to us by Farkas, this does not work. For simplicity, we consider a two-component degeneration  $X_0$  consisting of components  $Z_1, Z_2$ , each of genus 5, meeting at a node  $P$ , and the limit linear series having vanishing sequence 4, 5, 6, 7, 8 at  $P$  on each of the  $Z_i$ . On each component we then necessarily have the linear series  $(\mathcal{L}^i, V^i)$  obtained from the canonical linear series on  $Z_i$  by twisting by  $4P$ . Now, a limit of a hypothetical element  $\kappa$  of the kernel of (1.2) would yield tensors  $\kappa_i \in \text{Sym}^2 V^i$  on  $Z_i$ , each mapping to zero in  $\Gamma(Z_i, (\mathcal{L}^i)^{\otimes 2})$ , and having orders of vanishing at  $P$  adding up to at least 24 (here the order of vanishing of a tensor is defined to be the minimum of the orders of vanishing of each term in the tensor, assuming it has been written with respect to a basis of  $V^i$  with distinct orders of vanishing at  $P$ ). Now, each  $\text{Sym}^2 V^i$  is 15-dimensional, and we find that there is a 9-dimensional subspace of it spanned entirely by simple tensors vanishing to order at least 12 at  $P$ . Under the map  $\text{Sym}^2 V^i \rightarrow \Gamma(Z_i, (\mathcal{L}^i)^{\otimes 2})$ , the image of these subspaces are contained in  $\Gamma(Z_i, (\mathcal{L}^i)^{\otimes 2}(-12P))$ , which has dimension 8, so we conclude that there must be a nonzero element of the kernel. Thus,  $\kappa_i$  as described above do in fact exist.

An alternative point of view on the same example is to consider the limit linear series arising as the specialization of the image of the multiplication map, if we have a family of linear series specializing to the same limit linear series as above. If we follow the philosophy we have developed, but substitute usual Eisenbud-Harris limit linear series, we would attempt to show that the multiplication map has maximal rank generically by showing that the image of the multiplication map in the specialization can't be contained in limit linear series of dimension smaller than 15. However, in this case, it is clear that the image of (1.2) is contained in  $\Gamma(Z_i, (\mathcal{L}^i)^{\otimes 2}(-8P))$  on each component, which has dimension 12. Moreover, we can use  $(\mathcal{L}^i, \Gamma(Z_i, (\mathcal{L}^i)^{\otimes 2}(-8P)))$  to construct a crude Eisenbud-Harris limit linear series of degree 24 and dimension 11 on  $X_0$ , so in fact the multiplication map in the limit is contained in a smaller-dimensional limit linear series, and again, the approach fails.

This latter observation may appear to indicate that our approach should also fail in this case, since our global sections are glued from the sections arising in the above-described limit  $\mathfrak{g}_{24}^{11}$ . However, it is important that the limit linear series in question is crude, and in fact one can compute that in intermediate multidegrees, the space of global sections which can be obtained by gluing sections of  $(\mathcal{L}^1, \Gamma(Z_1, (\mathcal{L}^1)^{\otimes 2}(-8P)))$  to sections of  $(\mathcal{L}^2, \Gamma(Z_2, (\mathcal{L}^2)^{\otimes 2}(-8P)))$  has larger dimension. For instance, in multidegree (12, 12) the dimension of the space of such global sections is 15, meaning that our approach is not ruled out.

## 5. OBSERVATIONS ON INJECTIVITY

We now consider injective cases, meaning that  $\binom{r+m}{m} \leq md + 1 - g$ . Our main result will be the observation that any verification of the conditions of Theorem 3.9 for an injective case implies that we get infinitely many additional cases by increasing  $g$ . In fact, we will give two versions of this statement, with one adding a

mild hypothesis but yielding more cases in return. A preliminary definition is the following.

**Definition 5.1.** We say that a  $(g, r, d)$ -sequence  $\delta = (\delta_1, \dots, \delta_g)$  is **extendable** if for all  $g' \geq g$ , and all  $d'$  with  $g' \geq (r+1)(r+g-d')$ , we can extend  $\delta$  to a valid  $(g', r, d')$ -sequence.

Extendability occurs rather frequently. We have the following characterization:

**Proposition 5.2.** *A  $(g, r, d)$ -sequence  $\vec{\delta}$  is extendable if and only if 0 occurs at most one time more than  $r$  does in  $\vec{\delta}$ .*

*Proof.* First, if 0 occurs at most one time more than  $r$  does, then if we write  $\alpha = \lfloor g/(r+1) \rfloor$  and  $\beta = g - (r+1)\alpha$ , we must have  $0, 1, \dots, \beta-1$  appearing  $\alpha+1$  times, and  $\beta, \dots, r$  appearing  $\alpha$  times. Then for any  $g', d'$  as in the definition, we can extend  $\vec{\delta}$  to a  $(g', r, d')$ -sequence simply by adding  $g' - g$  entries which cycle through  $0, \dots, r+1$ , starting with  $\beta$ .

Conversely, if 0 occurs at least twice more than  $r$ , we first consider the case  $g' = g$ , and  $d'$  minimal with  $g \geq (r+1)(r+g-d')$  (i.e.,  $d' = r+g - \lfloor g/(r+1) \rfloor$ ). Then extendability would imply that  $r$  must occur at least  $r+g-d'$  times in  $\vec{\delta}$ , so 0 must occur at least  $r+g-d'+2$  times in  $\vec{\delta}$ . We also must have  $g > (r+1)(r+g-d')$  in order for 0 to occur more times, so in particular  $g$  is not a multiple of  $r+1$ . Next, consider  $g'' = (r+1)\lfloor g/(r+1) \rfloor \geq g$ , and  $d'' = r+g'' - \lfloor g/(r+1) \rfloor$ ; then in a  $(g'', r, d'')$ -sequence, every number must occur exactly  $r+g''-d''$  times, but  $r+g''-d'' = r+g-d'+1$ , and we already have 0 occurring at least  $r+g-d'+2$  times in  $\vec{\delta}$ , so there is no way to extend  $\vec{\delta}$  in this case.  $\square$

The following notion will be useful for verifying the steadiness condition in the  $m = 2$  case.

**Definition 5.3.** For a given  $m$ , we say  $w = (c_2, \dots, c_g) \in \mathbb{Z}^{g-1}$  is **unimaginative** if  $c_{i+1} - c_i \geq m$  for  $i = 2, \dots, g-1$ .

**Proposition 5.4.** *If  $w = (c_2, \dots, c_g)$  is unimaginative, then for any  $\vec{\delta}$  we have  $T(\vec{\delta})$  steady with respect to  $w$ .*

*Proof.* Indeed, if  $w' = (c'_2, \dots, c'_g) = (a_j^1, \dots, a_j^g)$ , then for any  $i < g$  we have

$$a_j^i \geq md - m - b_j^i = a_j^{i+1} - m,$$

so the sequence  $c'_i - c_i$  is nonincreasing, and  $(w', w)$  is steady.  $\square$

We have the following basic observation on ‘change of degree’:

**Proposition 5.5.** *Given  $(g, r, d, m)$ ,  $\vec{\delta}$ , and  $w$  as in Definition 3.6, if we have  $d' > d$  and a satisfying  $0 \leq a \leq d' - d$ , then  $\vec{\delta}$  is also an  $a$ -shifted  $(g, r, d')$ -sequence, and if we obtain  $w'$  from  $w$  by adding  $ma$  to every entry, then  $T_w(\vec{\delta})$  is  $N$ -expungeable for  $\vec{\delta}$  as a  $(g, r, d)$ -sequence if and only if  $T_{w'}(\vec{\delta})$  is  $N$ -expungeable for  $\vec{\delta}$  as an  $a$ -shifted  $(g, r, d')$ -sequence.*

*Proof.* Indeed,  $T_{w'}(\vec{\delta})$  is obtained from  $T_w(\vec{\delta})$  by adding  $ma$  to each  $a_j^i$ , and adding  $m(d' - d - a)$  to each  $b_j^i$ . One checks directly that the rules for expungeability are invariant under this operation.  $\square$

Below is our basic result on extending injective cases to higher genus.

**Proposition 5.6.** *Given  $(g, r, d, m)$  as in Definition 1.3, suppose that  $\binom{r+m}{m} \leq md+1-g$ , and suppose that there exists a  $(g, r, d)$ -sequence  $\vec{\delta}$  and a  $w = (c_2, \dots, c_g)$  such that  $T_w(\vec{\delta})$  is  $\binom{r+m}{m}$ -expungeable. Then for all  $(g', r, d', m)$  with  $g' \geq g$  and  $g' - d' \leq g - d$ , there exists a  $(g', r, d')$ -sequence  $\vec{\delta}'$  and a  $w' = (c'_2, \dots, c'_g)$  such that  $T_{w'}(\vec{\delta}')$  is  $\binom{r+m}{m}$ -expungeable. In particular, the Maximal Rank Conjecture holds in all these cases.*

*If further the above holds with  $\vec{\delta}$  extendable, then the condition  $g' - d' \leq g - d$  above is unnecessary.*

*Furthermore, in either situation, if the chosen  $w$  was unimaginative, then the new  $w$  may also be chosen to be unimaginative.*

*Proof.* Under either hypothesis, we have that  $\delta$  can be extended to a  $(g', r, d')$ -sequence  $\delta'$ : in the first case, the condition  $g' - d' \leq g - d$  allows us to extend simply by adding  $g' - g$  zeroes, while in the second case we can extend by hypothesis. Moreover, we have that  $\delta$  is a valid  $(g, r, d')$ -sequence, and Proposition 5.5 says that the  $\binom{r+m}{m}$ -expungeability of  $T_w(\vec{\delta})$  does not depend on whether we view  $\vec{\delta}$  as a  $(g, r, d)$ -sequence or a  $(g, r, d')$ -sequence (the only difference is that the  $b_j^i$  are all translated by  $m(d' - d)$ ). Then by appending sufficiently large (e.g., larger than  $md'$ ) numbers to  $w$ , we obtain  $w' \in \mathbb{Z}^{g'-1}$  with the property that  $T_{w'}(\vec{\delta}')$  is exactly the same as  $T_w(\vec{\delta})$ , when  $\vec{\delta}$  is considered as a  $(g, r, d')$ -sequence: the entries of  $T_{w'}(\vec{\delta}')$  after the first  $g$  columns are all erased. Then the  $\binom{r+m}{m}$ -expungeability of  $T_{w'}(\vec{\delta}')$  follows.

If  $w$  was unimaginative, the above construction can clearly also make  $w'$  unimaginative.  $\square$

We conclude by proving that for any fixed  $m, r$  we have injectivity for all  $g$  sufficiently large. Although the bound is very far from sharp (and is worse than that obtained in Larson [Lar12]), the proof is brief and we include it as an illustration of a different sort of approach to applying Theorem 3.9 than the ones which we will make below.

**Proposition 5.7.** *With  $m, r$  fixed, if we have  $g, d$  with  $\rho \geq 0$  and*

$$g \geq (r+1)((m+1)^{r-1} - r),$$

*then the Maximal Rank Conjecture holds for  $(g, r, d, m)$ . Moreover, a general chain of genus-1 curves is not in the closure of the locus on  $\mathcal{M}_g$  for which the maximal rank condition fails.*

*Proof.* We will show that with the stated lower bound, we can always produce a  $(g, r, d)$ -sequence  $\vec{\delta}$  so that for some column  $i_0$ , the entries  $a_j^{i_0}$  of  $T(\vec{\delta})$  are all distinct. Observe that this will be the case if the  $i_0$ th column of  $T'(\vec{\delta})$  is equal to  $0, 1, m+1, (m+1)^2, \dots, (m+1)^{r-1}$ , or more generally,  $a, a+1, a+m+1, a+(m+1)^2, \dots, a+(m+1)^{r-1}$  for some  $a$ . Thus, we take  $\vec{\delta}$  to be the sequence whose first  $(m+1)^{r-1} - r$  entries are 0, and then followed by  $(m+1)^{r-1} - (m+1)^{i-1} - (r-i)$  entries equal to  $i$ , for  $i = 1, \dots, r-1$ . We then take the next  $(m+1) - 2$  entries equal to 2, and then followed by  $(m+1)^{i-1} - i$  entries equal to  $i$  for  $i = 3, \dots, r$ . This determines the first  $(r+1)((m+1)^{r-1} - r)$  entries of  $\vec{\delta}$ , with each entry occurring

$(m+1)^{r-1} - r$  times. Any remaining entries of  $\vec{\delta}$  can be chosen to cycle from 0 through  $r$ .

Then set  $i_0$  to be the column immediately after the first sequence of  $(r-1)$ s occurring in  $\vec{\delta}$ , so that  $i_0 = r(m+1)^{r-1} - \binom{r}{2} - \sum_{i=1}^{r-2} (m+1)^i$ . By construction, the entries  $a_j^{i_0}$  of  $T'(\vec{\delta})$  have the desired form, so the entries  $a_j^{i_0}$  of  $T(\vec{\delta})$  are all distinct, as desired. Finally, let  $w = (c_2, \dots, c_g)$  with  $c_i = 0$  for  $i \leq i_0$  and  $c_i = md$  for  $i > i_0$ . Note that this is steady with respect to  $T(\vec{\delta})$  (indeed, for any  $\vec{\delta}$ ), and the effect is that every row occurs in the  $i_0$ th column of  $T_w(\vec{\delta})$ . We may then apply Rule (ii) repeated to  $T_w(\vec{\delta})$  to prove the desired statement.  $\square$

## 6. THE CASE OF QUADRICS

In this section, we use Theorem 3.9 to prove the Maximal Rank Conjecture for the  $m = 2$  case. The proof uses reduction constructions to show that we can always reduce either to smaller  $r$  or to one of a sequence of ‘critical’ cases which are in particular as close as possible to being simultaneously injective and surjective. Empirically, these critical cases are the most difficult cases to handle.

**Theorem 6.1.** *The Maximal Rank Conjecture holds in the  $m = 2$  case, and more specifically, for any given  $(g, r, d)$  with  $\rho \geq 0$  and  $r + g - d > 0$ , a general chain of genus-1 curves is not in the closure of the locus on  $\mathcal{M}_g$  where the maximal rank condition fails.*

*Explicitly, for every such  $(g, r, d)$  there is a  $(g, r, d)$ -sequence  $\vec{\delta}$  and an unimaginary  $w \in \mathbb{Z}^{g-1}$  such that  $T_w(\vec{\delta})$  is  $\min\left(\binom{r+2}{2}, 2d+1-g\right)$ -expungeable.*

In order to keep the overall structure of the proof as clear as possible, we will first state the necessary preliminary results, then give the proof of the theorem, and finally prove the preliminary results. In fact, we will also prove the statement of the theorem for many cases where  $r + g - d \leq 0$ , but to keep the statement as simple as possible we do not list precisely which cases are handled by our constructions.

The following lemma constitutes the basic reduction used for surjective cases.

**Lemma 6.2.** *Given  $(g, r, d)$  with  $\rho \geq 0$ , set  $m = 2$ , and write  $t = \min(\rho + (g + r - d), r - 1)$ . Setting  $r' = r - 1$ ,  $g' = g - t$  and  $d' = d - (t + 1)$  gives  $\rho' \geq 0$  and  $r' + g' - d' = r + g - d$ . Suppose that there is a  $(g', r', d')$ -sequence  $\vec{\delta}'$  having no more than  $r'$  of any given integer, and an unimaginary  $w' = (c'_2, \dots, c'_{g'}) \in \mathbb{Z}^{g'-1}$  such that  $T_{w'}(\vec{\delta}')$  is  $N$ -expungeable, and  $c'_2 \geq 2$ . Then there is a  $(g, r, d)$ -sequence  $\vec{\delta}$  having no more than  $r$  of any given integer, and an unimaginary  $w = (c_2, \dots, c_g) \in \mathbb{Z}^{g-1}$  such that  $T_w(\vec{\delta})$  is  $(N + t + 2)$ -expungeable and  $c_2 \geq 2$ .*

*In particular, if  $T_{w'}(\vec{\delta}')$  is  $(2d' + 1 - g')$ -expungeable, then  $T_w(\vec{\delta})$  is  $(2d + 1 - g)$ -expungeable. Moreover, if either  $\binom{r+2}{2} > 2d + 1 - g$  or  $\binom{r+2}{2} = 2d + 1 - g$  and  $\rho > 0$ , then  $\binom{r'+2}{2} \geq 2d' + 1 - g'$ .*

Thus, the reduction of the lemma can be applied to give lower bounds on rank in all cases, but the resulting bound may not be sharp unless we are starting in a surjective case which is either non-injective, or where  $\rho > 0$ . See Example 6.5 for further discussion.

We will also use Proposition 5.6 to reduce to the following sequence of ‘critical’ injective cases, of which the first was examined in Example 3.13 above.

**Proposition 6.3.** *Theorem 6.1 holds when  $r$  is even and  $g = (r+1)r/2$ ,  $d = (r+2)r/2$ , and when  $r$  is odd and  $g = (r+1)^2/2$ ,  $d = r(r+3)/2$ .*

Finally, the following computation is very straightforward, but is used in the proofs of both Theorem 6.1 and Lemma 6.2.

**Proposition 6.4.** *Given  $(g, r, d)$ , we have*

$$\binom{r+2}{2} - (2d+1-g) = \binom{r}{2} - \rho - (g+r-d)(r-1).$$

We can now complete the proof of the  $m = 2$  case of the Maximal Rank Conjecture.

*Proof of Theorem 6.1.* We work by induction on  $r$ , with the induction hypothesis being that Theorem 6.1 holds with the added stipulation that for any surjective case, we can arrange for the  $(g, r, d)$ -sequence  $\vec{\delta}$  to have at most  $r$  repetitions of every integer, and for  $w$  to be unimaginative, with  $c_2 \geq 2$ . We begin by proving the desired statement in the base case  $r = 3$ . The conditions  $\rho \geq 0$  and  $d < r+g$  imply that we must have  $g \geq r+1 = 4$ . We begin with the surjective cases, where  $\binom{r+2}{2} \geq 2d+1-g$ . By Proposition 6.4, this is equivalent to having  $3-\rho-(g+r-d) \cdot 2 \geq 0$ , implying that we must have  $g+r-d = 1$  and  $\rho \leq 1$ . Thus, the only two cases are the canonical case  $g = 4, d = 6$ , or the case  $g = 5, d = 7$ , which are addressed (satisfying our extra stipulations on the  $(g, r, d)$ -sequences and  $w$ ) in Examples 3.11 and 3.12. Now, the two previous cases are the only ones with  $g \leq 5$ , but we observe that Example 3.12 was injective, with  $\vec{\delta}$  extendable, so the  $r = 3$  case follows by Proposition 5.6.

Next, if we assume our hypothesis holds for  $r-1$ , Lemma 6.2 together with the induction hypothesis then gives us all surjective cases except for those which are also injective and have  $\rho = 0$ . Now, suppose that we are in the injective case  $\binom{r+2}{2} \leq 2d+1-g$ , and set  $s = \min(2d+1-g - \binom{r+2}{2}, \rho)$ . Then if we set  $g' = g-s, r' = r, d' = d-s$ , we see that  $g' + r' - d' = g + r - d$ , and  $\rho' = \rho - s \geq 0$ , so we have another valid case with the same  $r$ . In addition,

$$2d'+1-g' - \binom{r'+2}{2} = 2d+1-g - \binom{r+2}{2} - s = \max\left(0, 2d+1-g - \binom{r+2}{2} - \rho\right),$$

so  $(g', r', d')$  remains in the injective case, but either has  $\rho' = 0$ , or is simultaneously in the surjective case. In either case, Proposition 5.6 implies that in order to treat  $(g, r, d)$ , it is enough to treat  $(g', r', d')$ . Combined with our previous reductions in the surjective case, we see that it is enough to treat injective cases with  $\rho = 0$ . We claim that all such cases have  $g \geq (r+1)\lceil \frac{r}{2} \rceil$ . Indeed,  $\rho = 0$  means that  $g = (r+1)(g+r-d)$ , so it then suffices to see that injectivity (together with  $\rho = 0$ ) implies that  $g+r-d \geq r/2$ , which is immediate from Proposition 6.4. Noting that any  $(g, r, d)$ -sequence with  $\rho = 0$  is extendable, the theorem then follows from Propositions 6.3 and 5.6.  $\square$

We now give the proofs of the two intermediate results, starting with the basic reduction for the surjective case.

*Proof of Lemma 6.2.* It is immediate that  $r' + g' - d' = r + g - d$ , while we calculate that

$$\rho' = \rho - (t - (r + g - d)) = \rho - \min(\rho, d - g - 1) \geq 0.$$

We construct  $\vec{\delta}$  by adding 1 to each entry of  $\vec{\delta}'$ , and inserting  $t$  zeroes at the beginning of the sequence; since  $t \leq r - 1$ ,  $\vec{\delta}$  will have no number appearing more than  $r$  times. Moreover, we see that  $\vec{\delta}$  will be a  $(g, r, d)$ -sequence: since  $r + g - d = r' + g' - d'$ , this amounts to checking that no number in  $\vec{\delta}'$  appears more than  $t$  times. If  $t = r - 1$ , this is by hypothesis, while if  $t = \rho + (r + g - d)$ , this follows from the fact that  $\rho'$  is 0.

If  $w' = (c'_2, \dots, c'_{g'})$ , we construct  $w = (c_2, \dots, c_g)$  by setting  $c_2 = 3$ ,  $c_i = c_{i-1} + 2$  for  $i \leq t + 1$ , and  $c_i = c'_{i-t} + 2t + 2$  for  $i \geq t + 2$ . Then if  $w'$  is unimaginitive with  $c'_2 \geq 2$ , the same will be true of  $w$ . By construction we will have that in  $T_w(\vec{\delta})$ , only rows of the form  $(0, j_2)$  can appear in the first  $t$  columns: indeed, we have  $2d - c_{i+1} = 2d - 2i - 1$  while  $b_{(1,1)}^i = 2d - 2i - 2$  for  $i \leq t$ , so the  $(1, 1)$  row cannot appear, and  $b_{(j_1, j_2)}^i \leq b_{(1,1)}^i$  when  $j_1 \geq 1$ . Now, suppose there exists a choice of  $N$  rows of  $T_{w'}(\vec{\delta}')$  which can be used to verify  $N$ -expungeability of  $T_{w'}(\vec{\delta}')$ . Our claim is that using these rows (appropriately reindexed by 1 corresponding to the shift in  $\vec{\delta}$ ) together with the rows  $(0, 0), \dots, (0, t + 1)$ , we can verify  $(N + t + 2)$ -expungeability of  $T_w(\vec{\delta})$ . By construction we will have precisely the rows  $(0, 0), (0, 1), (0, 2)$  appearing in the first column, with entries  $a_j^1$  equal to 0, 1, 2 respectively, so repeatedly applying Rule (ii) we can drop these three rows. Next, in the following  $t - 1$  columns, we can have at most one new row appearing in each column, so applying Rule (iv) in each case, we can drop each of these rows, which are rows  $(0, 3), \dots, (0, t + 1)$ . Since we have already dropped  $(0, t + 2)$  up to  $(0, r)$ , the remaining rows are those of the form  $(j_1, j_2)$  with  $j_1 > 0$ , which appear only in the final  $g'$  columns. These  $g'$  columns of  $T_w(\vec{\delta})$  agree precisely with the  $T_{w'}(\vec{\delta}')$  one obtains from considering  $\vec{\delta}$  as a  $(t + 1)$ -shifted  $(g', r', d)$ -sequence, and the latter is  $N$ -expungeable by Proposition 5.5. We thus conclude the first statement of the lemma, and the particular case of  $(2d' + 1 - g')$ -expungeability follows immediately.

Finally, we verify that

$$\binom{r' + 2}{2} - (2d' + 1 - g') = \binom{r + 2}{2} - (2d + 1 - g) - (r - 1 - t)$$

by direct calculation, so for the last statement it suffices to prove that if  $t < r - 1$  and either  $\binom{r + 2}{2} > 2d + 1 - g$  or  $\binom{r + 2}{2} = 2d + 1 - g$  and  $\rho > 0$ , then  $\binom{r + 2}{2} - (2d + 1 - g) \geq r - 1 - t$ . Now, if  $t < r - 1$  then  $r - 1 - t = r - 1 - \rho - (r + g - d)$ . Writing  $\ell = r + g - d$ , Proposition 6.4 implies first that our desired inequality can be written  $\binom{r}{2} - \rho - \ell(r - 1) \geq r - 1 - \rho - \ell$ , and second, that under either of our hypotheses, we have  $\binom{r}{2} > \ell(r - 1)$ . The desired inequality simplifies to  $(r - 1)(r - 2)/2 \geq \ell(r - 2)$ , or equivalently,  $\ell \leq (r - 1)/2$ , while the given inequality yields  $\ell < r/2$  and hence  $\ell \leq (r - 1)/2$ , as desired.  $\square$

Finally, we treat our sequence of critical cases. Recall that an example of the  $r = 4$  case is given above in Example 3.13.

*Proof of Proposition 6.3.* Write  $\ell = g + r - d$ , so that  $\ell = \frac{r}{2}$  if  $r$  is even, and  $\ell = \frac{r+1}{2}$  if  $r$  is odd. Set

$$\vec{\delta} = \underbrace{0, \dots, 0}_{\ell \text{ times}}, \underbrace{1, \dots, 1}_{\ell \text{ times}}, \dots, \underbrace{r, \dots, r}_{\ell \text{ times}}.$$

We then set  $w = (c_2, \dots, c_g)$ , where  $c_2 = 2$ , and for  $2 < i \leq g/2 + 1$  we set

$$c_i = c_{i-1} + \begin{cases} 2: & i \not\equiv 2 \pmod{\ell} \\ 3: & i \equiv 2 \pmod{\ell}. \end{cases}$$

For  $g/2 + 1 < i \leq g$  we set

$$c_i = c_{i-1} + \begin{cases} 2: & i \not\equiv 1 \pmod{\ell} \\ 3: & i \equiv 1 \pmod{\ell}. \end{cases}$$

The result is that we have  $r + 1$  blocks consisting of  $\ell$  columns each, which can be analyzed essentially independently of one another. In addition, the situation is symmetric about the middle. Because our  $w$  is unimaginative, in order to analyze the erasures in  $T_w(\vec{\delta})$  we can simply look at how a given  $(a_j^i, b_j^i)$  compares to  $(c_i, 2d - c_{i+1})$ ; see Remark 3.14. Specifically, if  $\vec{j} = (j_1, j_2)$ , the columns are erased up until the first time that  $b_j^i \geq 2d - c_{i+1}$  (equivalently,  $a_j^{i+1} \leq c_{i+1}$ ), and will be erased after the last time that  $a_j^i \geq c_i$ . In particular, the  $(j_1, j_2)$  row appears for the first time in the  $i$ th column if and only if  $a_j^i > c_i$  and  $a_j^{i+1} \leq c_{i+1}$ .

Labeling our blocks  $0, \dots, r$ , we have the following formulas: if we write  $i = \ell \cdot \alpha + \beta$  with  $0 < \beta \leq \ell$ , so that the  $i$ th column of  $T(\vec{\delta})$  is the  $\beta$ th column of the  $\alpha$ th block, then provided that  $i \leq \frac{g}{2}$ , we have

$$c_i = 2i - 2 + \alpha - \delta_{\beta,1}, \text{ and } a_j^i = \begin{cases} i + j - 1: & \alpha < j \\ i + j - \beta: & \alpha = j \\ i + j - \ell - 1: & \alpha > j, \end{cases}$$

where  $\delta_{\beta,1}$  is the Kronecker  $\delta$  function. We then analyze which rows appear for the first time (reading left to right) in each column.

In the first column of the  $i$ th block, with  $0 \leq i < \ell$ , we will have the first appearances of the rows of the form  $(j, \ell + i - j)$ , for  $j = 0, \dots, i - 1$ . For the  $i'$ th column of the  $i$ th block, with  $1 < i' \leq i$ , the only new row is the  $(i, i)$  row, which occurs for the first time in the  $\lceil i/2 \rceil$ th column of the  $i$ th block (if  $i \leq 2$ , the  $(i, i)$  row occurs in the first column of the  $i$ th block). For  $i < i' \leq \ell$ , the row  $(i, i')$  will appear for the first time in the  $i'$ th column of the  $i$ th block (note that this includes the  $(0, 1)$  row occurring in the 1st column of the 0th block; for  $i > 0$ , we will have  $i' > 1$ ).

Now, if  $r$  even, the procedure we use to show that  $T_w(\vec{\delta})$  is  $\binom{r+2}{2}$ -expungeable is as follows: for  $i < r/2$ , we show that if all rows appearing in previous blocks have already been dropped, then we can work from left to right in the  $i$ th block to drop all rows appearing in that block. For  $i > r/2$  we apply the same procedure from right to left, and finally in the central  $r/2$  block, we have dropped all rows appearing in any other block, and we show that the rows only appearing in the  $r/2$  can be dropped as well.

The desired dropping behavior is clear in the 0th block, since according to the above description, we see that when we work from left to right, there are never more than two new rows appearing in a given column, so repeated use of Rule (iv) suffices to drop all rows. The same argument works for the 1st block. In the  $i$ th block for  $1 < i < r/2$ , we have at most  $i + 1$  new rows appearing in the first column:  $(0, i + r/2), (1, i + r/2 - 1), \dots, (i - 1, r/2 + 1)$  always appear, as well as  $(i, i)$  when



$i = 2$ . However, in the next  $i - 1$  columns we have no new rows appearing other than  $(i, i)$  in the  $\lceil i/2 \rceil$ th column, and in each subsequent column we have only one new row appearing. We claim that we can use Rule (vii) with  $n = i$  to drop the  $i + 1$  rows appearing in the first  $i$  columns; this will then imply that the rest of the rows in the block can be dropped just using Rule (iv), as in the 0th block. Now, within the  $i$ th block, the rows  $(0, i + r/2), (1, i + r/2 - 1), \dots, (i - 1, r/2 + 1)$  are all identical, starting at  $(2i(r/2) + i, 2d - 2i(r/2) - i - 2)$ , with the left side increasing by 2 and the right side decreasing by 2 in each subsequent column. Note that this precisely matches the behavior of  $w$ , so in fact these rows all appear throughout the  $i$ th block. In contrast, the  $(i, i)$ th row is a constant  $(2i(r/2 + 1), 2d - 2i(r/2 + 1))$ , and appears in the  $\lceil i/2 \rceil$ th column only if  $i$  is odd, and in the  $\lceil i/2 \rceil$ th and  $(\lceil i/2 \rceil + 1)$ st columns if  $i$  is even. Because no other rows appear in these columns, we can apply Rule (vii), as claimed.

By symmetry, we can also work from right to left to drop all rows except those which occur solely in the  $r/2$  block. But these rows are precisely the rows  $(0, r), (1, r - 1), \dots, (r/2, r/2)$ , and we can again apply Rule (vii), this time with  $n = r/2$ , to drop all the remaining rows. This handles the case that  $r$  is even.

Next, if  $r$  is odd, the situation is almost the same, except that the number of blocks is even. Accordingly, we can drop all rows by first going from left to right in the first  $(r + 1)/2$  blocks, and then going right to left in the remaining  $(r + 1)/2$  blocks. We again have that the 0th and first blocks each have at most two new rows in each column, so we can eliminate all the rows simply using Rule (iv). We also still have that the  $i$ th block for  $i \leq (r + 1)/2$  will have  $i + 1$  rows occurring in the first  $i$  columns, and then one additional row in each subsequent column, so just as before, we can apply Rule (vii) to treat the first  $i$  columns of the block simultaneously, and then Rule (iv) to deal with the remaining columns. As before, the situation is symmetric, so applying the same procedure from right to left on the remaining  $(r + 1)/2$  blocks will allow us to drop all rows, as desired.  $\square$

**Example 6.5.** We consider some examples of the reduction processes from the proof of Theorem 6.1.

First, if we have the canonical case, with  $g = r + 1$  and  $d = 2r$ , then applying Lemma 6.2 we have  $\rho = 0$  and  $g + r - d = 1$ , so  $t = 1$ , and we get  $r' = r - 1$ ,  $g' = g - 1 = r' + 1$ ,  $d' = d - 2 = 2r'$ . Thus, we reduce to the canonical case in genus one less.

Next, suppose we have an injective case with  $r$  even and  $g$  strictly smaller than the critical case  $\frac{r(r+1)}{2}$ . Then our reduction process will lead to an injective (and surjective) case with  $r' = r - 1$ , and  $g'$  strictly smaller than the critical case  $\frac{(r'+1)^2}{2}$ . However, the next step in the reduction will not necessarily stay below the critical case. For instance, consider the case  $r = 6, g = 20, d = 24$ . This is injective, with  $\rho = 6$  and  $g + r - d = 2$ , and  $2d + 1 - g - \binom{r+2}{2} = 1$ . In this case, the  $s$  from the proof of Theorem 6.1 is equal to 1, so we first use Proposition 5.6 to reduce to considering the case  $r' = r = 6, g' = g - 1 = 19, d' = d - 1 = 23$ . This case is now injective and surjective, with  $g' + r' - d' = 2$  and  $\rho' = 5$ , so when we apply Lemma 6.2, we have  $t = r' - 1 = 5$ , and reduce to the case  $r'' = r' - 1 = 5, g'' = g' - 5 = 14, d'' = d' - 6 = 17$ , which is still an injective and surjective case, and has  $g'' = 14 < \frac{(r''+1)^2}{2} = 18$ . The next step is another reduction via Lemma 6.2, where now we have  $t = r'' - 1 = 4$ , so the next reduction ends up at the critical

case  $r''' = 4$ ,  $g''' = 10$ ,  $d''' = 12$ , which is addressed directly in Proposition 6.3 (and in Example 3.13).

Finally, consider what happens for the critical case  $r = 4$ ,  $g = 10$ ,  $d = 12$  if instead of handling the case directly as in our proof of Theorem 6.1, we instead attempt to apply Lemma 6.2. This case has  $g + r - d = 2$  and  $\rho = 0$ , so we will have  $t = 2$ , so we will ‘reduce’ to the case  $r' = 3$ ,  $d' = 9$ ,  $g' = 8$ . However, this latter case is non-surjective:  $2d' + 1 - g' = 11$ , while  $\binom{r'+2}{2} = 10$ . Thus, the best we can do in this case is to show that we have rank 10 for  $(g', r', d') = (8, 9, 3)$ . Then Lemma 6.2 says that we have rank at least  $10 + t + 2 = 14$  for  $(g, r, d) = (10, 4, 12)$ , but the conjecture is that this case should have rank 15. Thus, in this case Lemma 6.2 does provide partial information, but falls short of the sharp result.

## 7. OBSERVATIONS ON SURJECTIVITY

We now consider the surjective range, where  $\binom{r+m}{m} \geq md + 1 - g$ . We prove surjectivity in a range of cases for  $m = 3$  in Corollary 7.5 below, but while these cases are somewhat different from those considered by Jensen and Payne in [JP], they are fully covered by Ballico [Bal12a]. For us, the purpose of this section is to illustrate a rather distinct type of argument from that found in other sections, and simultaneously to explain how the number  $md + 1 - g$ , which arises naturally from the Riemann-Roch theorem on smooth curves, can be seen also in the context of limit linear series and our elementary criterion. We start our discussion with the limit linear series point of view, but this will not be used elsewhere: the criteria which we will actually apply are stated in Proposition 7.3 below, and proved directly from our elementary criterion.

Suppose we have  $w = (c_2, \dots, c_g)$  inducing multidegree  $(d_1, \dots, d_g)$ , with  $\sum_i d_i = md$ . Then we can study  $\Gamma(X_0, \mathcal{L}_{\text{md}(w)})$  via the Riemann-Roch theorem for reducible curves, but for our purposes, it is more instructive to carry out a direct analysis. Considering restriction to components and nodes gives us an exact sequence

$$(7.1) \quad 0 \rightarrow \Gamma(X_0, \mathcal{L}_{\text{md}(w)}) \rightarrow \bigoplus_{i=1}^g \Gamma(Z_i, \mathcal{L}_{\text{md}(w)}|_{Z_i}) \rightarrow \bigoplus_{i=1}^{g-1} k,$$

and assuming all the  $d_i$  are positive, we have  $\dim \Gamma(Z_i, \mathcal{L}_{\text{md}(w)}|_{Z_i}) = d_i$  for  $i = 1, \dots, g$ . We thus see that  $\dim \Gamma(X_0, \mathcal{L}_{\text{md}(w)}) \geq md + 1 - g$ , with equality if and only if the last map of (7.1) is surjective. We then have

**Proposition 7.1.** *In the above situation, suppose that  $md > 2g - 2$ , and we have  $d_1 \geq 1$ ,  $d_i \geq 2$  for  $1 < i < g$ , and  $d_g \geq 1$ . Then (7.1) is surjective, so  $\dim \Gamma(X_0, \mathcal{L}_{\text{md}(w)}) = md + 1 - g$ .*

*Proof.* Since  $md > 2g - 2$ , there is some  $i_0$  for which the above inequality on  $d_{i_0}$  becomes strict. If  $1 < i_0 < g$ , and  $d_{i_0} > 2$ , then the map  $\Gamma(Z_{i_0}, \mathcal{L}_{\text{md}(w)}|_{Z_{i_0}}) \rightarrow k^{\oplus 2}$  induced by restriction to  $P_{i_0}$  and  $Q_{i_0}$  is necessarily surjective. For  $1 < i < i_0$ , because  $d_i \geq 2$  we have surjectivity of the map  $\Gamma(Z_i, \mathcal{L}_{\text{md}(w)}|_{Z_i}) \rightarrow k$  induced by restriction to  $P_i$ , and similarly for  $i < i_0 < g$  we have surjectivity of the map  $\Gamma(Z_i, \mathcal{L}_{\text{md}(w)}|_{Z_i}) \rightarrow k$  induced by restriction to  $Q_i$ . Putting these together gives surjectivity of (7.1). A similar analysis of the cases  $i_0 = 1$  and  $i_0 = g$  yields the proposition.  $\square$

*Remark 7.2.* The hypothesis in Proposition 7.1 that  $md > 2g - 2$  is quite mild: if  $m = 3$ , it is always satisfied, while for  $m = 2$ , we observe that if we are in

the surjective range, so that  $\binom{r+2}{2} \geq 2d+1-g$ , then we necessarily have  $d > g$ . Indeed, Proposition 6.4 may be rewritten equivalently as  $\binom{r+2}{2} - (2d+1-g) = (d-g)(r-1) - \binom{r}{2} - \rho$ , from which  $d > g$  follows immediately when the lefthand side is nonnegative.

The above point of view leads to the observation that we can prove surjectivity by studying spans, without necessarily choosing sections to drop and then proving linear independence. For instance, if reading from left to right, the first column has full  $d_1$ -dimensional span, and each subsequent column has full  $(d_i-1)$ -dimensional span among the sections not appearing in previous columns, then we obtain surjectivity.

We now restate the above observation in the context of our elementary criterion (and in a somewhat generalized form), and derive some consequences. In the below, the  $i_0 = 1$  case corresponds to the above situation; it turns out that it is convenient to generalize to the situation that the first several columns of  $T_w(\vec{\delta})$  may be empty, allowing us to then omit some values of  $a_j^i$ .

**Proposition 7.3.** *In the situation of Theorem 3.9, suppose that  $\binom{r+m}{m} \geq md+1-g$ , write  $\text{md}(w) = (d_1, \dots, d_g)$ , and suppose that  $d_i > 0$  for  $i > 1$ . Suppose also that for some  $i_0 \geq 1$ , we have  $\sum_{i=1}^{i_0} (d_i - 1) \geq 0$ , and there is some choice of  $md+1-g$  rows of  $T_w(\vec{\delta})$  such that after initially dropping the other  $\binom{r+m}{m} - (md+1-g)$  rows, we can apply Rules (ii)-(v) of Definition 3.7 to first drop  $1 + \sum_{i=1}^{i_0} (d_i - 1)$  rows of the  $i_0$ th column of  $T_w(\vec{\delta})$ , and then for each  $i > i_0$ , to drop  $d_i - 1$  rows of the  $i$ th column of  $T_w(\vec{\delta})$ , none of which occur in previous columns. Then  $T_w(\vec{\delta})$  is  $(md+1-g)$ -expungeable.*

*In particular, suppose that  $w$  and  $i_0$  are as above, and  $T_w(\vec{\delta})$  has the property that the non-erased portion of each row is contiguous. Then if every number between 0 and  $md$  other than  $1, \dots, i_0 - 1$  and  $md - 1$  occurs among the  $a_j^i$  of  $T_w(\vec{\delta})$  for  $i \geq i_0$ , we have that  $T_w(\vec{\delta})$  is  $(md+1-g)$ -expungeable.*

*More generally, if  $w$  and  $i_0$  are as above, and  $T_w(\vec{\delta})$  has the property that the non-erased portion of each row is contiguous, suppose further that:*

- *in the  $i_0$ th column, either  $0, i_0, i_0 + 1, \dots, c_{i_0+1} - 1$  all occur among the  $a_j^{i_0}$ , or  $0, i_0, i_0 + 1, \dots, c_{i_0+1} - 2$  all occur, with  $c_{i_0+1} - 2$  occurring at least twice;*
- *for each  $i > i_0$ , in the  $i$ th column either  $c_i + 1, \dots, c_{i+1} - 1$  all occur among the  $a_j^i$ , or  $c_i + 1, \dots, c_{i+1} - 2$  all occur, with  $c_{i+1} - 2$  occurring at least twice.*

*Then  $T_w(\vec{\delta})$  is  $(md+1-g)$ -expungeable.*

Note that the condition on the non-erased portion of each row being contiguous is automatically satisfied for unimaginative  $w$ , or more generally for  $w$  which are steady with respect to  $T(\vec{\delta})$ .

*Proof.* The hypothesis of the first statement is just a special form of  $(md+1-g)$ -expungeability, since  $1 + \sum_{i=1}^g (d_i - 1) = md+1-g$ .

For the second statement, we observe that a number  $a$  can occur as  $a_j^i$  in  $T_w(\vec{\delta})$  only if we have  $c_i \leq a \leq c_{i+1}$  (here, we take  $c_1 = 0$  and  $c_{g+1} = md$ ): certainly, we must have  $a \geq c_i$ , but we must likewise have  $b_j^i \geq md - c_{i+1}$ , and because

$a_j^i + b_j^i \leq md$ , we also obtain  $a \leq c_{i+1}$ . Now, we will denote by  $S$  the set of  $N$  rows chosen to verify  $N$ -expungeability, which we will construct one column at a time.

By hypothesis, we have  $c_i < c_{i+1}$  for all  $i > 1$ , so we see that if any of  $0, \dots, c_{i_0+1} - 1$  occur among the  $a_j^i$  in the  $i$ th column with  $i \geq i_0$ , we must have  $i = i_0$ . We have supposed that  $c_{i_0+1} - (i_0 - 1) (= 1 + \sum_{i=1}^{i_0} (d_i - 1))$  of these values do occur, so we can choose  $S$  to contain exactly one row with each of these values in the  $i_0$ th column. Then, if we use Rule (i) to drop the rows not in  $S$  initially, we can apply Rule (ii) to drop the remaining  $1 + \sum_{i=1}^{i_0} (d_i - 1)$  rows in this column. Then for  $i > i_0$ , the values  $c_i + 1, \dots, c_{i+1} - 1$  can only occur in the  $i$ th column. Moreover, if  $c_i + 1 \leq a_j^i$ , then the  $j$ th row cannot occur in a previous column, since  $a_j^i > c_i$  implies that the row cannot appear in the  $(i - 1)$ st column, and we have assumed that the nonerased portions of each row are contiguous. Thus, we may again add rows to  $S$  so that the  $i$ th column contains each value from  $c_i + 1$  to  $c_{i+1} - 1$  exactly once, and we can again apply Rule (ii) to drop  $c_{i+1} - c_i - 1 = d_i - 1$  rows from the  $i$ th column. Note that by construction, the number of rows in  $S$  is precisely  $1 + \sum_{i=1}^g (d_i - 1) = md + 1 - g$ , and applying the first statement of the proposition, we conclude the desired result.

Finally, the more general case proceeds by exactly the same argument, except that in columns where  $c_{i+1} - 1$  is omitted, but  $c_{i+1} - 2$  occurs at least twice, we use Rule (iv) to drop the final two rows in the column.  $\square$

**Example 7.4.** Consider the canonical series, with  $r = g - 1$  and  $d = 2g - 2$ . In this case, the only  $(g, r, d)$ -sequence is  $\vec{\delta} = 0, 1, \dots, r$ , and then we have that the  $i$ th column of  $T'(\vec{\delta})$  is:

$$\begin{array}{cc} i-2 & d-i+1 \\ i-1 & d-i \\ \vdots & \vdots \\ 2i-4 & d-2i+3 \\ 2i-2 & d-2i+2 \\ 2i-1 & d-2i \\ \vdots & \vdots \\ i+r-1 & d-i-r \end{array}$$

In formulas,  $a_j^i = j + i - 1 - \gamma_{i,j}$ , where  $\gamma_{i,j}$  is 1 if  $j = 0, \dots, i - 2$ , and is 0 if  $j = i - 1, \dots, r$ , and  $b_j^i = d - 1 - a_j^i$  for  $j \neq i - 1$ , and  $b_{i-1}^i = d - a_{i-1}^i$ .

Now, for any  $m \geq 2$ , we form  $T(\vec{\delta})$  by adding  $m$ -tuples of rows of  $T'(\vec{\delta})$ , and we then set  $w = (c_2, \dots, c_g)$ , with  $c_i = a_{(i-2, \dots, i-2, r)}^i$  for all  $i$ . We have  $c_{i+1} - c_i = 2(m - 1) + 1$  for all  $i$ .

It is then straightforward to verify the following (see also the first paragraph of the proof of Corollary 7.5 below). First, the rows  $(0, \dots, 0, j)$  for  $0 \leq j \leq r$  all appear in the first column of  $T_w(\vec{\delta})$ , and the corresponding values of  $a_j^1$  are  $0, 1, \dots, r = c_2 - 1$ . Next, in the  $i$ th column for  $1 < i < g = r + 1$ , rows of the form  $\vec{j} = (i - 2, \dots, i - 2, i - 1, \dots, i - 1, j)$  with  $j = r - 1$  or  $r$  all appear in  $T_w(\vec{\delta})$ , except for  $(i - 2, \dots, i - 2, r - 1)$ . The corresponding values of  $a_j^i$  yield  $c_i, c_i + 1, \dots, c_{i+1} - 1$ . Finally, in the  $g$ th column, the rows  $(j_1, r - 1, \dots, r - 1, r, \dots, r)$  with  $j_1 = r - 2$  or  $r - 1$  and  $j_m = r$  all appear with the exception of  $(r - 2, r - 1, \dots, r - 1, r)$  (which

has  $a_j^i = c_g - 1$ , and the values of  $a_j^i$  these yield cover  $c_g, c_g + 1, \dots, md - 2$ . Then the row  $(r, \dots, r)$  has  $a_j^i = md$ , and (the  $i_0 = 1$  case of) Proposition 7.3 gives us surjectivity.

We now apply Proposition 7.3 to prove surjectivity within certain ranges, generalizing the canonical linear series, but including many cases which do not fall in the surjective range for  $m = 2$ . Recall from the introduction that although we only treat the case  $m = 3$  directly, surjectivity then follows for all higher  $m$ .

**Corollary 7.5.** *Suppose that  $m = 3$ , and  $(g, r, d)$  satisfy  $\rho \geq 0$ . The then Maximal Rank Conjecture holds in the following cases:*

- (i) *if  $r + g - d = 1$ , and  $2r - 3 \geq \rho + 1$ ;*
- (ii) *if  $r + g - d = 2$ ,  $r \geq 4$ , and  $2r - 3 \geq \rho + 2$ .*

*Moreover, the locus of chains of genus-1 curves is not in the closure of the locus in  $\mathcal{M}_g$  where the maximal rank condition fails.*

*Proof.* We begin with some general observations, which apply for any  $m$ . Suppose that we have  $w = (c_2, \dots, c_g)$ , and that the  $c_i$  are nondecreasing. Then in the  $i$ th column, each  $a_j^i$  is at least  $c_i$ . Thus, if we want every number to appear as some  $a_j^{i'}$  in  $T_w(\vec{\delta})$ , we need  $c_i - 1$  to appear before the  $i$ th column. In most cases, it will have to appear in the  $(i - 1)$ st column, and if we have  $c_i - 1 = a_j^{i-1}$  for some  $\vec{j}$ , then we must also have  $b_j^{i-1} \geq md - c_i$ . Since  $a_j^{i-1} + b_j^{i-1}$  is given by  $md - m$  plus the number of times  $\delta_{i-1}$  occurs in  $\vec{j}$ , we conclude that  $\delta_{i-1}$  must occur at least  $m - 1$  times in  $\vec{j}$ . Similarly, if  $c_i - n$  appears as  $a_j^{i-1}$  for  $1 \leq n < m$ , we conclude that we must have  $\delta_{i-1}$  occurring at least  $m - n$  times in  $\vec{j}$ . If  $w$  is unimaginate, we then derive a necessary and sufficient condition for numbers of the form  $c_i - n$  to appear as  $a_j^{i-1}$  in  $T_w(\vec{\delta})$  for some  $\vec{j}$ : first, we must have  $c_i - n \geq c_{i-1}$ , second,  $c_i - n$  must appear as some  $a_j^{i-1}$  in  $T(\vec{\delta})$ , and third, if  $n < m$ , it must do so in a row  $\vec{j}$  with at least  $m - n$  occurrences of  $\delta_{i-1}$  in  $\vec{j}$ .

In case (i), we set  $\vec{\delta}$  to be the sequence whose first  $\rho$  entries are 0, followed by  $0, 1, \dots, r$ , and set  $n = \min(r - 1, \rho + 2)$ . Then set  $w = (c_2, \dots, c_g)$ , where we set  $c_i = -3(\rho + 3 - n - i) - 1$  for  $2 \leq i \leq \rho + 3 - n$ , and  $c_i = a_{j_i}^i$  for the remaining  $i$ , with:

- $\vec{j}_{\rho+2-i} = (0, n - i, n - i)$  for  $0 \leq i \leq n - 2$ ;
- $\vec{j}_{\rho+i} = (i - 2, i - 2, r)$  for  $3 \leq i \leq r + 1$ ;

We have that  $w$  is unimaginate: indeed,  $c_i - c_{i-1} = 3$  for  $2 < i \leq \rho + 3 - n$ ,  $c_{\rho+4-n} - c_{\rho+3-n} = 1 + 2(\rho + 5 - n)$ ,  $c_{\rho+2-i} - c_{\rho+1-i} = 4$  for  $0 \leq i < n - 2$ ,  $c_{\rho+3} - c_{\rho+2} = (2(\rho + 2) + r + \rho + 2) - 2(\rho + 1 + n) = \rho + r - 2n + 4$ ,  $c_{\rho+i} - c_{\rho+i-1} = 5$  for  $3 < i \leq r + 1$ , and one sees from our definition of  $n$  that  $1 + 2(\rho + 5 - n) > 3$  and  $\rho + r - 2n + 4 \geq 3$ .

With this setup, it is not hard to see that  $T_w(\vec{\delta})$  will satisfy the condition of Proposition 7.3. Specifically, no rows will appear in the first  $\rho + 2 - n$  columns. Next, we use the inequality  $2r - 3 \geq \rho + 1$  to conclude that  $r \geq \rho + 3 - n$ , and then in the  $(\rho + 3 - n)$ th column, rows of the form  $(0, 0, j_3)$  with  $0 \leq j \leq \rho + 3 - n$  will yield  $a_j^i$  equal to  $0, \rho + 3 - n, \rho + 4 - n, \dots, 2(\rho + 3 - n) - 1$ . Then the rows  $(0, 1, 1)$ ,  $(0, 1, 2)$ ,  $(0, 2, 2)$ ,  $(0, 1, 3)$  give  $2(\rho + 3 - n)$ ,  $2(\rho + 3 - n) + 1$ , and  $2(\rho + 4 - n)$

twice. We thus have the numbers 0 through  $2(\rho + 4 - n)$  occurring with  $\rho + 2 - n$  gaps in this column, and with  $2(\rho + 4 - n)$  occurring twice. Then in the  $\rho + 2 - i$ th column for  $i = n - 2, \dots, 1$ , we will have the rows  $(0, n - i, n - i)$ ,  $(0, n - i, n + 1 - i)$ ,  $(0, n + 1 - i, n + 1 - i)$  and  $(0, n - i, n + 2 - i)$  contributing  $2(\rho + 1 + n - 2i)$ ,  $2(\rho + 1 + n - 2i) + 1$ , and  $2(\rho + 2 + n - 2i)$  twice. In each case, we will have skipped  $c_{\rho+2-i} - 1 = 2(\rho + 1 + n - 2i) - 1$ , but we can still apply Proposition 7.3 because  $2(\rho + 1 + n - 2i) - 2$  will have appeared twice in the previous column.

Next, in the  $(\rho + 2)$ nd column, the rows  $(1, 1, j_3)$  for  $1 \leq j_3 \leq r$  cover all values from  $\max(3(\rho + 2), c_{\rho+2})$  to  $c_{\rho+3} - 1$ . If  $3(\rho + 2) \leq c_{\rho+2}$ , these rows suffice in this column, and otherwise, we must have  $n = r - 1$ , and the hypothesis  $2r - 3 \geq \rho + 1$  implies that  $c_{\rho+2} \geq 3(\rho + 2) - 2$ , so adding in the rows  $(0, r - 1, r - 1)$  and  $(0, r - 1, r)$  allows us to cover all values between  $c_{\rho+2}$  and  $c_{\rho+3} - 1$ . In the  $(\rho + i)$ th column for  $i = 3, \dots, r$ , the rows  $(i - 2, i - 2, r)$ ,  $(i - 2, i - 1, r - 1)$ ,  $(i - 2, i - 1, r)$ ,  $(i - 1, i - 1, r - 1)$ ,  $(i - 1, i - 1, r)$  give the values from  $c_{\rho+i}$  to  $c_{\rho+i+1} - 1$ . Finally, in the  $(\rho + r + 1)$ st column, the rows  $(r - 1, r - 1, r)$ ,  $(r - 2, r, r)$ ,  $(r - 1, r, r)$ ,  $(r, r, r)$  give the values from  $c_{\rho+r+1}$  to  $3d$ , skipping only  $3d - 1$ . Applying Proposition 7.3, we conclude the desired statement for case (i).

For case (ii), the pattern is similar, but a bit more complicated. We set  $\vec{\delta}$  to be the sequence whose first  $\rho$  entries are 0, followed by  $0, 0, 1, 1, \dots, r, r$ . For  $\rho > 0$ , set  $n = \min(r - 1, \rho + 1)$ , and set  $n = 2$  if  $\rho = 0$ . Then set  $w = (c_2, \dots, c_g)$ , where we set  $c_i = -3(\rho + 4 - n - i) - 1$  for  $2 \leq i \leq \rho + 4 - n$ , and  $c_i = a_{j_i}^i$  for the remaining  $i$ , with:

- $\vec{j}_{\rho+3-i} = (0, n - i, n - i)$  for  $0 \leq i \leq n - 2$ ;
- $\vec{j}_{\rho+2i} = (i - 1, i - 1, r - 2)$  for  $2 \leq i \leq r - 2$ ;
- $\vec{j}_{\rho+2i+1} = (i - 1, i - 1, r)$  for  $2 \leq i \leq r - 1$ ;
- $\vec{j}_{\rho+2r-2} = (r - 3, r - 2, r)$ ;
- $\vec{j}_{\rho+2r} = (r - 2, r - 1, r - 1)$ ;
- $\vec{j}_{\rho+2r+1} = (r - 3, r - 1, r)$ ; and
- $\vec{j}_{\rho+2r+2} = (r - 2, r, r)$ .

We have that  $w$  is unimaginate: indeed,  $c_i - c_{i-1} = 3$  for  $2 < i \leq \rho + 4 - n$ ,  $c_{\rho+5-n} - c_{\rho+4-n} = 1 + 2(\rho + 6 - n)$ ,  $c_{\rho+3-i} - c_{\rho+2-i} = 4$  for  $0 \leq i \leq n - 3$ ,  $c_{\rho+4} - c_{\rho+3} = (2(\rho + 3) + r + \rho + 1) - 2(\rho + 2 + n) = \rho + r - 2n + 3$ ,  $c_{\rho+2i+1} - c_{\rho+2i} = 3$  for  $2 \leq i \leq r - 2$ ,  $c_{\rho+2i} - c_{\rho+2i-1} = 5$  for  $3 \leq i \leq r - 2$ ,  $c_{\rho+2r-2} - c_{\rho+2r-3} = 5$ ,  $c_{\rho+2r-1} - c_{\rho+2r-2} = 3$ ,  $c_{\rho+2r} - c_{\rho+2r-1} = 3$ ,  $c_{\rho+2r+1} - c_{\rho+2r} = 3$ ,  $c_{\rho+2r+2} - c_{\rho+2r+1} = 5$ , and one sees from our definition of  $n$  that  $\rho + r - 2n + 3$  is always at least 3 (using also that  $r \geq 4$  for the  $\rho = 0$  case).

We again verify that  $T_w(\vec{\delta})$  will satisfy the condition of Proposition 7.3. Specifically, no rows will appear in the first  $\rho + 3 - n$  columns. Next, we use the inequalities  $2r - 3 \geq \rho + 2$  and  $r \geq 4$  to conclude that  $r \geq \rho + 4 - n$ , and then in the  $(\rho + 4 - n)$ th column, rows of the form  $(0, 0, j_3)$  with  $0 \leq j_3 \leq \rho + 4 - n$  will include  $a_{j_3}^i$  equal to  $0, \rho + 4 - n, \rho + 5 - n, \dots, 2(\rho + 4 - n) - 1$ . Then the rows  $(0, 1, 1)$ ,  $(0, 1, 2)$ ,  $(0, 2, 2)$ ,  $(0, 1, 3)$  give  $2(\rho + 4 - n)$ ,  $2(\rho + 4 - n) + 1$ , and  $2(\rho + 5 - n)$  twice. We thus have the numbers 0 through  $2(\rho + 5 - n)$  occurring with  $\rho + 3 - n$  gaps in this column, and with  $2(\rho + 5 - n)$  occurring twice. Then in the  $\rho + 3 - i$ th column for  $i = n - 2, \dots, 1$ , we will have the rows  $(0, n - i, n - i)$ ,  $(0, n - i, n + 1 - i)$ ,  $(0, n + 1 - i, n + 1 - i)$  and  $(0, n - i, n + 2 - i)$  contributing  $2(\rho + 2 + n - 2i)$ ,  $2(\rho + 2 + n - 2i) + 1$ , and  $2(\rho + 3 + n - 2i)$  twice. In each case, we will have skipped

$c_{\rho+3-i} - 1 = 2(\rho + 2 + n - 2i) - 1$ , but we can still apply Proposition 7.3 because  $2(\rho + 2 + n - 2i) - 2$  will have appeared twice in the previous column.

Next, in the  $(\rho + 3)$ rd column, the rows  $(1, 1, j_3)$  for  $1 \leq j_3 \leq r - 2$  cover all values from  $\max(3(\rho + 3), c_{\rho+3})$  to  $c_{\rho+4} - 1$ . If  $3(\rho + 3) \leq c_{\rho+3}$ , these rows suffice in this column, and otherwise, the hypothesis  $2r - 3 \geq \rho + 2$  implies that adding the rows  $(0, n, n)$  and  $(0, n, n + 1)$  suffices to cover all values from  $c_{\rho+3}$  up to  $3(\rho + 3) - 1$ . In the  $(\rho + 2i)$ nd column for  $i = 2, \dots, r - 2$ , the rows  $(i - 1, i - 1, r - 2)$ ,  $(i - 1, i - 1, r - 1)$  and  $(i - 1, i - 1, r)$  give the values from  $c_{\rho+2i}$  to  $c_{\rho+2i+1} - 1$ . In the  $(\rho + 2i + 1)$ st column for  $i = 2, \dots, r - 2$ , the rows  $(i - 1, i - 1, r)$ ,  $(i - 1, i, r - 2)$ ,  $(i - 1, i, r - 1)$ ,  $(i - 1, i, r)$ ,  $(i, i, r - 2)$  give the values from  $c_{\rho+2i+1}$  to  $c_{\rho+2i+2} - 1$ . We have to change the pattern slightly in the final five columns, as follows: in the  $(\rho + 2r - 2)$ nd column, the final row of the previous column was  $(r - 2, r - 2, r - 2)$ , but this does not appear in the  $(\rho + 2r - 2)$ nd column, because  $c_{\rho+2r-2}$  was chosen to be one larger than the corresponding  $a_j^i$ . Instead,  $c_{\rho+2r-2}$  will be achieved by the  $(r - 3, r - 2, r)$  row, and then the  $(r - 2, r - 2, r - 1)$  and  $(r - 2, r - 2, r)$  rows cover through  $c_{\rho+2r-1} - 1$ . In the  $(\rho + 2r - 1)$ st column, the rows  $(r - 2, r - 2, r)$ ,  $(r - 3, r - 1, r - 1)$ ,  $(r - 2, r - 1, r - 1)$  cover from  $c_{\rho+2r-1}$  to  $c_{\rho+2r} - 1$ . In the  $(\rho + 2r)$ th column, the rows  $(r - 2, r - 1, r - 1)$ ,  $(r - 3, r - 1, r)$ ,  $(r - 1, r - 1, r - 1)$  cover from  $c_{\rho+2r}$  to  $c_{\rho+2r+1} - 1$ . In the  $(\rho + 2r + 1)$ st column, the rows  $(r - 3, r - 1, r)$ ,  $(r - 2, r - 1, r)$ ,  $(r - 1, r - 1, r)$ ,  $(r - 3, r, r)$ ,  $(r - 2, r, r)$  cover from  $c_{\rho+2r+1}$  to  $c_{\rho+2r+2} - 1$ , and in the final column, the rows  $(r - 2, r, r)$ ,  $(r - 1, r, r)$ ,  $(r, r, r)$  will cover from  $c_{\rho+2r+2}$  to  $3d$ , omitting only  $3d - 1$ . Applying Proposition 7.3, we conclude the desired statement for case (ii).  $\square$

*Remark 7.6.* Although Corollary 7.5 is generally quite far from being sharp, the  $r \geq 4$  condition in case (ii) is in fact necessary: otherwise, we have that  $r = 3$ ,  $g = 9$ ,  $d = 10$  satisfies the hypotheses of the corollary, but this case is not even in the surjective range.

*Remark 7.7.* Similar arguments should give surjectivity results also when  $r + g - d > 2$ , but they are more complicated than the cases treated in Corollary 7.5. Moreover, it appears that any arguments using Proposition 7.3 will not prove any cases of surjectivity which are not in the surjective range already for  $m = 3$ ; the reason comes from the first paragraph of the proof of Corollary 7.5, which shows that if we want to skip at most one value of the  $a_j^i$  in going from one column to the next, strict conditions on the choices of  $w$  follow, which prevent us from taking advantage of larger  $m$ . However, it is possible that if we modified Proposition 7.3 to impose restrictions on direction of approach, as discussed in Remark 4.12, we would be able to take advantage of the flexibility provided by larger values of  $m$ . Alternatively, one could generalize Proposition 7.3 to take into account for instance Rule (vii) of Definition 3.7, which should likewise allow us to take better advantage of large  $m$ .

## 8. THE CASE OF CUBICS

We conclude with a discussion of the  $m = 3$  case. Rather than attempting to prove that it holds for every case of given small  $r$ , which requires extensive case-by-case analysis, we will treat what appear to be the “hardest” cases for each of  $r = 3, 4, 5$ , each of which is in the injective range, and then conclude by Proposition 5.6 that the Maximal Rank Conjecture holds for all but finitely many cases for each  $r$ . The aforementioned “hardest case” for each  $r$  is somewhat parallel to the critical

cases for  $m = 2$  addressed in Proposition 6.3; specifically, we take the smallest  $g$  such that all non-injective cases occur in genera strictly smaller than  $g$ . For  $r = 5$ , this case happens to be also in the surjective range. For  $r = 3$  and  $r = 4$  these cases are not in the surjective range, although the  $r = 3$  example will imply a case having genus one greater which is simultaneously in the injective and surjective ranges.

The three examples are as follows.

**Example 8.1.** Consider the case  $r = 3$ ,  $g = 7$ ,  $d = 9$ . Then  $\binom{r+3}{3} = 20$ , and  $3d + 1 - g = 21$ ; we see that this is in the injective range. We take the extendable  $(g, r, d)$ -sequence  $\vec{\delta} = 0, 0, 1, 1, 2, 2, 3$ , which gives  $T'(\vec{\delta})$  as follows.

0	9	0	9	0	8	1	7	2	6	3	5	4	4
1	7	2	6	3	6	3	6	3	5	4	4	5	3
2	6	3	5	4	4	5	3	6	3	6	3	6	2
3	5	4	4	5	3	6	2	7	1	8	0	9	0

We then get  $T(\vec{\delta})$  as follows.

	23	4	20	7	17	10	14	13	10	17	6	21		
(0, 0, 0)	0	27	0	27	0	24	3	21	6	18	9	15	12	12
(0, 0, 1)	1	25	2	24	3	22	5	20	7	17	10	14	13	11
(0, 0, 2)	2	24	3	23	4	20	7	17	10	15	12	13	14	10
(0, 1, 1)	2	23	4	21	6	20	7	19	8	16	11	13	14	10
(0, 0, 3)	3	23	4	22	5	19	8	16	11	13	14	10	17	8
(0, 1, 2)	3	22	5	20	7	18	9	16	11	14	13	12	15	9
(1, 1, 1)	3	21	6	18	9	18	9	18	9	15	12	12	15	9
(0, 1, 3)	4	21	6	19	8	17	10	15	12	12	15	9	18	7
(0, 2, 2)	4	21	6	19	8	16	11	13	14	12	15	11	16	8
(1, 1, 2)	4	20	7	17	10	16	11	15	12	13	14	11	16	8
(0, 2, 3)	5	20	7	18	9	15	12	12	15	10	17	8	19	6
(1, 1, 3)	5	19	8	16	11	15	12	14	13	11	16	8	19	6
(1, 2, 2)	5	19	8	16	11	14	13	12	15	11	16	10	17	7
(0, 3, 3)	6	19	8	17	10	14	13	11	16	8	19	5	22	4
(1, 2, 3)	6	18	9	15	12	13	14	11	16	9	18	7	20	5
(2, 2, 2)	6	18	9	15	12	12	15	9	18	9	18	9	18	6
(1, 3, 3)	7	17	10	14	13	12	15	10	17	7	20	4	23	3
(2, 2, 3)	7	17	10	14	13	11	16	8	19	7	20	6	21	4
(2, 3, 3)	8	16	11	13	14	10	17	7	20	5	22	3	24	2
(3, 3, 3)	9	15	12	12	15	9	18	6	21	3	24	0	27	0
	23	4	20	7	17	10	14	13	10	17	6	21		

The highlighted entries show  $T_w(\vec{\delta})$  for  $w = (4, 7, 10, 13, 17, 21)$ , which is unimagnative. As in earlier examples, we have placed the  $c_i$  and  $md - c_i$  at the top and bottom of the table to make the erasure procedures clearer.

Now, by applying Rule (ii) to the first, third, fourth and seventh columns, we can drop rows  $(0, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 1, 2)$ ,  $(0, 1, 3)$ ,  $(1, 1, 1)$ ,  $(1, 1, 2)$ ,  $(1, 1, 3)$ ,  $(2, 2, 3)$ ,  $(0, 3, 3)$ ,  $(1, 3, 3)$ ,  $(2, 3, 3)$  and  $(3, 3, 3)$ . Applying Rule (iii) to the sixth column, we can also drop rows  $(1, 2, 3)$ ,  $(0, 2, 3)$ , and  $(2, 2, 2)$ . This leaves only five rows, which can all be dropped using Rule (iv) in the second, first and fifth columns.

**Example 8.2.** Consider the case  $r = 4$ ,  $g = 16$ ,  $d = 17$ . Then  $\binom{r+3}{3} = 35$ , and  $3d + 1 - g = 36$ , so this is in the injective range, but not the surjective range. We



take the extendable  $(g, r, d)$ -sequence  $\vec{\delta} = 0, 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3$ , which gives  $T'(\vec{\delta})$  as follows.

0	17	0	17	0	17	0	17	0	16	1	15	2	14	3	13	4	12	5	11	6	10	7	9	8	8	9	7	10	6	11	5
1	15	2	14	3	13	4	12	5	12	5	12	5	12	5	11	6	10	7	9	8	8	9	7	10	6	11	5	12	4	13	3
2	14	3	13	4	12	5	11	6	10	7	9	8	8	9	7	10	6	11	5	12	4	13	3	14	2	1	5	12	4	13	3
3	13	4	12	5	11	6	10	7	9	8	8	9	7	10	6	11	5	12	4	13	3	14	2	1	5	12	4	13	3	14	2
4	12	5	11	6	10	7	9	8	8	9	7	10	6	11	5	12	4	13	3	14	2	1	5	12	4	13	3	14	2	1	5

We then get  $T(\vec{\delta})$  as follows.

(0,0,0)	0	48	3	46	5	44	7	39	12	35	16	32	19	29	22	27	24	23	28	20	31	16	35	14	37	10	41	7	44	4	47	
(0,0,1)	1	49	2	48	3	47	4	46	5	44	7	42	9	40	11	37	14	34	17	31	20	28	23	25	26	22	29	19	32	16	35	13
(0,0,2)	2	48	3	47	4	46	5	45	6	42	9	39	12	36	15	34	17	32	19	30	21	27	24	24	27	21	30	18	33	15	36	12
(0,0,3)	3	47	4	46	5	45	6	44	7	41	10	38	13	35	16	32	19	29	22	26	25	24	27	22	29	20	31	17	34	14	37	11
(0,1,2)	3	46	5	44	7	42	9	40	11	38	13	36	15	34	17	32	19	30	21	28	23	25	26	22	29	19	32	16	35	13	38	10
(1,1,1)	3	45	6	42	9	39	12	36	15	36	15	36	15	36	15	33	18	30	21	27	24	24	27	21	30	18	33	15	36	12	39	9
(0,0,4)	4	46	5	45	6	44	7	43	8	40	11	37	14	34	17	31	20	28	23	25	26	22	29	19	32	16	35	14	37	12	39	10
(0,1,3)	4	45	6	43	8	41	10	39	12	37	14	35	16	33	18	30	21	27	24	24	27	22	29	20	31	18	33	15	36	12	39	9
(0,2,2)	4	45	6	43	8	41	10	39	12	36	15	33	18	30	21	29	22	28	23	27	24	24	27	21	30	18	33	15	36	12	39	9
(1,1,2)	4	44	7	41	10	38	13	35	16	34	17	33	18	32	19	30	21	28	23	26	25	23	28	20	31	17	34	14	37	11	40	8
(0,1,4)	5	44	7	42	9	40	11	38	13	36	15	34	17	32	19	29	22	26	25	23	28	20	31	17	34	14	37	12	39	10	41	8
(0,2,3)	5	44	7	42	9	40	11	38	13	35	16	32	19	29	22	27	24	25	26	23	28	21	30	19	32	17	34	14	37	11	40	8
(1,1,3)	5	43	8	40	11	37	14	34	17	33	18	32	19	31	20	28	23	25	26	22	29	20	31	18	33	16	35	13	38	10	41	7
(1,2,2)	5	43	8	40	11	37	14	34	17	32	19	30	21	28	23	27	24	26	25	25	26	22	29	19	32	16	35	13	38	10	41	7
(0,3,4)	6	43	8	41	10	39	12	37	14	34	17	31	20	28	23	25	26	22	29	19	32	18	33	17	34	16	35	13	38	10	41	7
(1,1,4)	6	42	9	39	12	36	15	33	18	32	19	31	20	30	21	27	24	24	27	21	30	18	33	15	36	12	39	10	41	8	43	6
(1,2,3)	6	42	9	39	12	36	15	33	18	31	20	29	22	27	24	25	26	23	28	21	30	19	32	17	34	15	36	12	39	9	42	6
(2,2,2)	6	42	9	39	12	36	15	33	18	30	21	27	24	24	27	24	27	24	27	21	30	18	33	15	36	12	39	9	42	6	43	5
(0,3,4)	7	42	9	40	11	38	13	36	15	33	18	30	21	27	24	24	27	21	30	18	33	16	35	14	37	12	39	10	41	8	43	6
(1,2,4)	7	41	10	38	13	35	16	32	19	30	21	28	23	26	25	24	27	22	29	20	31	17	34	14	37	11	40	9	42	7	44	5
(1,3,3)	7	41	10	38	13	35	16	32	19	30	21	28	23	26	25	23	28	20	31	17	34	16	35	15	36	14	37	11	40	8	43	5
(2,2,3)	7	41	10	38	13	35	16	32	19	29	22	26	25	23	28	22	29	21	30	20	31	18	33	16	35	14	37	11	40	8	43	5
(0,4,4)	8	41	10	39	12	37	14	35	16	32	19	29	22	26	25	23	28	20	31	17	34	14	37	11	40	8	43	7	44	6	45	5
(1,3,4)	8	40	11	37	14	34	17	31	20	29	22	27	24	25	26	22	29	19	32	16	35	14	37	12	39	10	41	8	43	6	45	4
(2,2,4)	8	40	11	37	14	34	17	31	20	28	23	25	26	22	29	21	30	20	31	19	32	16	35	13	38	10	41	8	43	6	45	4
(2,3,3)	8	40	11	37	14	34	17	31	20	28	23	25	26	22	29	20	31	18	33	16	35	15	36	14	37	13	38	10	41	7	44	4
(1,4,4)	9	39	12	36	15	33	18	30	21	28	23	26	25	24	27	21	30	18	33	15	36	12	39	9	42	6	45	5	46	4	47	3
(2,3,4)	9	39	12	36	15	33	18	30	21	27	24	27	21	30	19	32	17	34	15	36	13	38	11	40	9	42	7	44	5	46	3	
(3,3,3)	9	39	12	36	15	33	18	30	21	27	24	24	27	21	30	18	33	15	36	12	39	12	39	12	39	12	39	9	42	6	45	3
(2,4,4)	10	38	13	35	16	32	19	29	22	26	25	23	28	20	31	18	33	16	35	14	37	11	40	8	43	5	46	4	47	3	48	2
(3,3,4)	10	38	13	35	16	32	19	29	22	26	25	23	28	20	31	17	34	14	37	11	40	10	41	9	42	8	43	6	45	4	47	2
(3,4,4)	11	37	14	34	17	31	20	28	23	25	26	22	29	19	32	16	35	13	38	10	41	8	43	6	45	4	47	3	48	2	49	1
(4,4,4)	12	36	15	33	18	30	21	27	24	24	27	21	30	18	33	15	36	12	39	9	42	6	45	3	48	0	51	0	51	0	51	0
	48	3	46	5	44	7	39	12	35	16	32	19	29	22	27	24	23	28	20	31	16	35	14	37	10	41	7	44	4	47		

The highlighted entries show  $T_w(\vec{\delta})$  for

$$w = (3, 5, 7, 12, 16, 19, 22, 24, 28, 31, 35, 37, 41, 44, 47).$$

Note that this  $w$  is not unimaginative, although one can check that it is still steady with respect to  $T(\vec{\delta})$ .

We may use Rules (ii) and (iii) to drop all rows in the first, second, third, sixth, seventh, eighth, 10th and 16th columns. We can also drop the rows in the fourth column with Rule (ii) and Rule (iv). This leaves two rows in each of the fifth and ninth columns, which can thus be dropped with Rule (iv). The remaining rows in the 11th column can then be dropped with Rule (ii), and the remaining rows in the 15th column can then be dropped with Rule (iii). This leaves only one row in the 12th column and two rows in the 14th column, so these can be dropped with Rule (iv). Finally, this leaves only one row in the 13th column, so we are done.

**Example 8.3.** Consider the case  $r = 5$ ,  $g = 26$ ,  $d = 27$ . Then  $\binom{r+3}{3} = 56$ , and  $3d + 1 - g = 56$ , so this is in both the injective and surjective ranges. We take the extendable  $(g, r, d)$ -sequence  $\vec{\delta} = 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5$ , which gives  $T'(\vec{\delta})$  as follows.

27	27	0	27	0	27	0	27	0	26	1	25	2	24	3	23	4	22	5	21	6	20	7	19	8	18	9	17	10	16	11	15	12	14	13	13	14	12	15	11	16	10	17	9	18	8	19	7	20	6
25	25	2	24	3	23	4	22	5	21	6	21	6	21	6	21	6	20	7	19	8	18	9	17	10	16	11	15	12	14	13	13	14	12	15	11	16	10	17	9	18	8	19	7	20	6	21	5		
2	24	3	23	4	22	5	21	6	20	7	19	8	18	9	17	10	16	11	15	12	15	12	15	12	15	12	14	13	13	14	12	15	11	16	10	17	9	18	8	19	7	20	6	21	5	22	4	23	3
3	23	4	22	5	21	6	20	7	19	8	18	9	17	10	16	11	15	12	14	13	13	14	12	15	11	16	10	17	9	18	8	19	7	20	6	21	5	22	4	23	3	24	2	25	1	26	0		
4	22	5	21	6	20	7	19	8	18	9	17	10	16	11	15	12	14	13	13	14	12	15	11	16	10	17	9	18	8	19	7	20	6	21	5	22	4												
5	21	6	20	7	19	8	18	9	17	10	16	11	15	12	14	13	13	14	12	15	11	16	10	17	9	18	8	19	7	20	6	21	5	22	4														

$$w = (3, 6, 8, 11, 15, 18, 21, 24, 28, 31, 34, 37, 41, 44, 47, 50, 53, 56, 60, 63, 66, 70, 73, 76, 78).$$

Now, first observe that the first, fourth, 17th, 24th, 25th and 26th columns can have all their rows dropped using Rule (iii), and the 18th column can have its rows dropped with Rule (ii). We can then drop the remaining row in the third column, and then the remaining row in the second column. Likewise, we can drop the two remaining rows in the 23rd column. Next, in the 22nd column we have the rows (2, 4, 5), (3, 3, 5) and (4, 4, 4) remaining, with  $\delta_{22} = 4$ . Then by Rule (v) with  $j = 5$  we can drop the (4, 4, 4) row, and then by Rule (iv) the two remaining rows. We again use Rule (iv) to drop the two remaining rows in the 21st column, and then again in the 20th column, and then the 19th column (recalling that we have already dropped the rows in the 18th column). Thus, it remains to consider the rows which only appear in the fifth through 16th columns.

Next, note that in the sixth column, the rows appearing are  $(0, 1, 4)$ ,  $(0, 1, 5)$ ,  $(0, 2, 3)$  and  $(1, 1, 1)$ . Since  $\delta_6 = 1$ , we can apply Rule (v) with  $j = 0$  to drop row  $(1, 1, 1)$ . But then the remaining rows in the seventh column are  $(0, 1, 5)$ ,  $(0, 2, 3)$  and  $(1, 1, 2)$ , and  $\delta_7 = 1$ , so using Rule (v) with  $j = 2$  we can drop the  $(0, 1, 5)$  row, and this allows us to drop the two remaining rows in the seventh column, and subsequently in the sixth, fifth and eighth columns, by using Rule (iv) repeatedly. Similarly, the rows appearing in the 10th column are  $(0, 2, 4)$ ,  $(0, 3, 3)$ ,  $(1, 2, 2)$  and  $(1, 2, 3)$ , and  $\delta_{10} = 1$ , so we can use Rule (v) with  $j = 2$  to drop row  $(0, 3, 3)$ . The remaining rows in the ninth column are then  $(0, 2, 4)$ ,  $(1, 1, 5)$  and  $(1, 2, 2)$ , and  $\delta_9 = 1$ , so we can again apply Rule (v) with  $j = 2$  to drop  $(1, 1, 5)$ , and then Rule (iv) to drop the remaining rows in the ninth, and subsequently the 10th, 11th and 12th columns. We can apply the same procedure a third time to the 13th and 14th columns, first using Rule (v) in the 13th column with  $j = 3$  to drop row  $(2, 2, 4)$ , and then again using Rule (v) with  $j = 3$ , but this time in the 14th column, to drop row  $(2, 2, 5)$ . We can then use Rule (iv) to drop the remaining rows in the 13th and 14th columns.

We are left with the rows supported only in the 15th and 16th columns, which are  $(0, 3, 5)$ ,  $(1, 3, 4)$  and  $(2, 3, 3)$ . We can finally drop these using Rules (vi) and (iv).

Combining Examples 8.1, 8.2 and 8.3 with Proposition 5.6, we conclude:

**Corollary 8.4.** *The Maximal Rank Conjecture holds for  $m = 3$ , and*

- (i)  $r = 3$  with  $g \geq 7$ ;
- (ii)  $r = 4$  with  $g \geq 16$ ;
- (iii)  $r = 5$  with  $g \geq 26$ ;

*Moreover, in these cases the locus of  $\overline{\mathcal{M}}_g$  consisting of chains of genus-1 curves is not in the closure of the locus in  $\mathcal{M}_g$  where the appropriate maximal rank condition fails.*

Note that (subject to the hypothesis  $r + g - d > 0$ ) Corollary 7.5 covers all  $m = 3$  cases with  $r = 3$  and  $g \leq 6$ , with  $r = 4$  and  $g \leq 9$ , and with  $r = 5$  and  $g \leq 12$  (as well a number of additional cases). Thus, there are no missing cases for  $r = 3$ , and a relatively small number for  $r = 4$ , but a rather significant number for  $r = 5$ . We expect that any given one of these cases can be handled as above, but do not see any simple way of handling them all simultaneously.

*Remark 8.5.* Comparing to previously known injectivity results for  $m = 3$ , Larson [Lar12] obtains injectivity for  $r = 3$  when  $g \geq 9$ , for  $r = 4$  when  $g \geq 19$ , and for  $r = 5$  when  $g \geq 35$ . Jensen and Payne [JP] obtain all cases for  $r = 3$  and  $r = 4$ , but in  $r = 5$  only treat the case  $\rho = 0$ , which translates to  $g \geq 30$ .

*Remark 8.6.* It is interesting to note that for  $r = 4$  and  $g = 14$ , all cases are injective; in fact, the smallest allowable  $d$ , which is  $d = 16$ , gives a case which is both injective and surjective. However, the case  $g = 15$ ,  $d = 16$  is not injective, which is why we are forced to start with  $g = 16$  above. Indeed, the noninjective genus-15 case, together with Proposition 5.6, imply that we cannot treat the  $g = 14$ ,  $d = 16$  case with any extendable  $(g, r, d)$ -sequence (however, it is not difficult to treat with a non-extendable sequence).

## REFERENCES

- [AF11] Marian Aprodu and Gavril Farkas, *Koszul cohomology and applications to moduli*, Grassmannians, moduli spaces, and vector bundles (Emma Previato, ed.), Clay Mathematics Proceedings, vol. 14, 2011, pp. 25–50.
- [Bal09] Edoardo Ballico, *On the maximal rank conjecture in  $\mathbb{P}^4$* , International Journal of Pure and Applied Mathematics **53** (2009), no. 3, 363–376.
- [Bal12a] ———, *Embeddings of general curves in projective spaces in the range of the cubics*, International Journal of Pure and Applied Mathematics **81** (2012), no. 4, 613–617.
- [Bal12b] ———, *Embeddings of general curves in projective spaces: the range of the quadrics*, Lithuanian Mathematical Journal **52** (2012), no. 2, 134–137.
- [BE87a] Edoardo Ballico and Philippe Ellia, *Beyond the maximal rank conjecture for curves in  $\mathbb{P}^3$* , Space curves (Rocca di Papa, 1985), Lecture Notes in Mathematics, vol. 1266, Springer, 1987, pp. 1–23.
- [BE87b] ———, *The maximal rank conjecture for nonspecial curves in  $\mathbb{P}^n$* , Mathematische Zeitschrift **196** (1987), no. 3, 355–367.
- [EH83a] David Eisenbud and Joe Harris, *Divisors on general curves and cuspidal rational curves*, Inventiones Mathematicae **74** (1983), 371–418.
- [EH83b] ———, *A simpler proof of the Gieseker-Petri theorem on special divisors*, Inventiones Mathematicae **74** (1983), 269–280.

- [EH86] ———, *Limit linear series: Basic theory*, *Inventiones Mathematicae* **85** (1986), no. 2, 337–371.
- [EH87] ———, *Irreducibility and monodromy of some families of linear series*, *Annales scientifiques de l'École Normale Supérieure* **20** (1987), no. 1, 65–87.
- [Far09] Gavril Farkas, *Koszul divisors on moduli spaces of curves*, *American Journal of Mathematics* **131** (2009), no. 3, 819–867.
- [FL81] William Fulton and Robert Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*, *Acta Mathematica* **146** (1981), 271–283.
- [FO11] Gavril Farkas and Angela Ortega, *The maximal rank conjecture and rank two Brill-Noether theory*, *Pure and Applied Mathematics Quarterly* **7** (2011), no. 4, 1265–1295, Special issue in honor of Eckart Viehweg.
- [GH80] Phillip Griffiths and Joseph Harris, *On the variety of special linear systems on a general algebraic curve*, *Duke Mathematical Journal* **47** (1980), 233–272.
- [Gie82] David Gieseker, *Stable curves and special divisors: Petri's conjecture*, *Inventiones Mathematicae* **66** (1982), 251–275.
- [JP] David Jensen and Sam Payne, *Combinatorial and inductive methods for the tropical maximal rank conjecture*, preprint.
- [JP16] ———, *Tropical independence II: The maximal rank conjecture for quadrics*, *Algebra and Number Theory* **10** (2016), no. 8, 1601–1640.
- [Lar12] Eric Larson, *The maximal rank conjecture for sections of curves*, preprint, 2012.
- [Oss06] Brian Osserman, *A limit linear series moduli scheme*, *Annales de l'Institut Fourier* **56** (2006), no. 4, 1165–1205.
- [Oss14] ———, *Limit linear series moduli stacks in higher rank*, preprint, 2014.
- [Tei03] Montserrat Teixidor i Bigas, *Injectivity of the symmetric map for line bundles*, *Manuscripta Mathematica* **112** (2003), no. 4, 511–517.
- [Voi92] Claire Voisin, *Sur l'application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri*, *Acta Mathematica* **168** (1992), no. 1, 249–272.